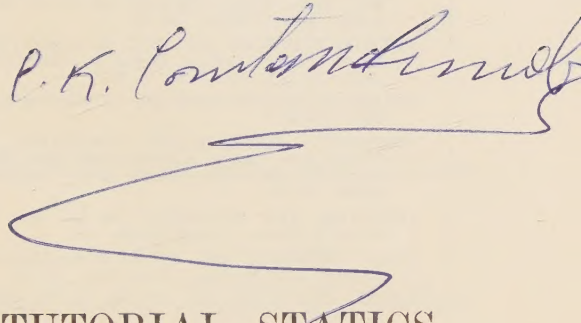
STATICS

Digitized by the Internet Archive
in 2025

https://archive.org/details/bwb_KU-357-526



THE TUTORIAL STATICS

SOME TEXTBOOKS IN

Mathematics and Science.

ALGEBRA, THE TUTORIAL. By Wm. Briggs, LL.D., D.C.L., M.A., B.Sc., F.C.S., F.R.A.S., and G. H. Bryan, Sc.D., F.R.S. *Fourth Edition.* 8s. 6d.

BOTANY, TEXTBOOK OF. By J. M. Lowson, M.A., B.Sc., F.L.S. *Seventh Edition, Revised, in collaboration with the Author,* by L. C. Fox, M.A. 9s. 6d.

CALCULUS, GROUNDWORK OF. By William Hunter, M.A., B.Sc. 5s. 6d.

CHEMISTRY, INTERMEDIATE INORGANIC. By G. H. Bailey, D.Sc., and D. R. Snellgrove, Ph.D., M.Sc., F.C.S., A.I.C. Volume I. Non-Metals. 6s. 6d. Volume II. Mainly Metals. 6s. 6d.

DYNAMICS, THE TUTORIAL. By Dr. Wm Briggs and Dr. G. H. Bryan. *Second Edition.* 5s.

HEAT TEXTBOOK OF. By R. W. Stewart, D.Sc., and Professor John Satterly, D.Sc., M.A. 6s. 6d.

LIGHT, TEXTBOOK OF. By Dr. R. W. Stewart, and Professor John Satterly. *Sixth Edition.* 6s. 6d.

ELECTRICITY AND MAGNETISM, INTERMEDIATE TEXTBOOK OF. By R. W. Hutchinson, M.Sc. *Second Edition.* 9s. 6d.

ORGANIC CHEMISTRY, THE ELEMENTS OF. By E. I. Lewis, M.A., B.Sc. *Second Edition.* 4s.

PROPERTIES OF MATTER. By C. J. L. Wagstaff, M.A. *Fourth Edition.* 5s.

QUALITATIVE ANALYSIS. By Dr. Wm. Briggs and Dr. R. W. Stewart. 4s.

QUANTITATIVE ANALYSIS, ELEMENTARY. By Wm. Briggs, LL.D. M.A., B.Sc., F.C.S., and H. W. Bausor, M.A., F.C.S., F.I.C. 3s.

SOUND, TEXTBOOK OF. By Edmund Catchpool, B.Sc. *Fifth Edition.* 6s. 6d.

STATICS, THE TUTORIAL. By Dr. Wm. Briggs and Dr. G. H. Bryan. *Fourth Edition.* 5s.

TRIGONOMETRY, THE TUTORIAL. By Dr. Wm. Briggs and Dr. G. H. Bryan. *Third Edition.* 5s.

ZOOLOGY, TEXTBOOK OF. By H. G. Wells, B.Sc., and A. M. Davies, D.Sc. *Seventh Edition.* 8s. 6d.

London: W. B. CLIVE

University Tutorial Press Ld.

High St., New Oxford St., W.C.

THE TUTORIAL STATICS

BY

WILLIAM BRIGGS, LL.D., M.A., B.Sc., F.R.A.S.

AND

G. H. BRYAN, Sc.D., F.R.S.

LATE HONORARY FELLOW OF ST. PETER'S COLLEGE, CAMBRIDGE, AND
PROFESSOR OF MATHEMATICS IN THE UNIVERSITY COLLEGE OF NORTH WALES

Fourth Edition (*Fourteenth Impression*)

LONDON : W. B. CLIVE

University Tutorial Press Ltd.

HIGH ST., NEW OXFORD ST., W.C.

1930

PRINTED IN GREAT BRITAIN BY UNIVERSITY TUTORIAL PRESS LD., AT THE
BURLINGTON PRESS, FOXTON, NEAR CAMBRIDGE,

PREFACE TO THE THIRD EDITION.

In its present form this book deals thoroughly with all those portions of the subject which can be satisfactorily treated without the use of higher analytical methods. A knowledge of the elements of Trigonometry has been assumed, as this is practically essential to the subject excepting in its most elementary stages. Also graphic methods are introduced and fully discussed.

In each chapter will be found a certain number of typical problems fully solved. These are intended to show the student the various methods of work at his disposal; but at the same time it is advisable that the student should attempt the problem for himself before reading the solution in the book. The Examples for practice are very numerous and vary considerably in difficulty; but a sufficient course of work for the average student is indicated by the list of "Selected Problems" given at the head of each set of Examples. This course avoids the most difficult problems, and selects from the rest a sufficient number to give the necessary amount of practice. Where, however, the student finds a difficulty in any type of problem it will be advisable for him to work others of that type which are not included in the selected list. The harder problems will give excellent practice to the more ambitious.

Two sizes of type are used, all the important bookwork being printed in the larger type, while hints, explanations, examples, alternative proofs, and a few of the less important theorems are printed in smaller type. The more fundamental propositions—such as the Parallelogram of Forces—have their *numbers* as well as their headings printed in

the dark type (thus **135**). These the student should be able to reproduce from memory.

The " Summaries of Results " at the end of each chapter are intended to facilitate a final revision of the whole subject, but they must only be used when the text has been *thoroughly* mastered.

Where results are given as *corollaries*, it is not in most cases sufficient in proving them to quote the *result* of the proposition on which they depend ; the student will do well to write out complete proofs, employing, as far as desirable, the same *methods* of proof as are used for the propositions themselves.

In the letterpress letters which denote *points* in figures are printed thus—**P, Q,** so that **PQ** will denote a line and **PQR** an angle or a triangle. Letters denoting algebraic *magnitude*, such as forces, are printed in ordinary italics thus—*P, Q,* so that *PQ* and *PQR* denote products of two and three algebraic quantities respectively. In a few of the *figures* this rule has not been adhered to where no confusion is likely to arise.

It is hoped that this book will be found especially suitable to students preparing for London University Examinations, as (without using either Analytical Geometry or the Calculus) it deals with all the subjects required at the Intermediate Science and Final B.A. Examinations.

The alterations and substantial additions effected in this edition are the work of Mr. A. G. Cracknell, M.A., B.Sc., F.C.P.

NOTE TO THE FOURTH EDITION.

In this edition the book has been entirely reset, but no alteration of any moment has been made in the matter, and the pagination remains practically the same as in the Third Edition.

CONTENTS.

PAGES.

CHAPTER I.—THE PARALLELOGRAM OF FORCES. Representation of forces by straight lines. Proof and experimental verification of Parallelogram of Forces. Triangle and Polygon of Forces; their converses and applications. Resultant of two perpendicular forces 1—23

CHAPTER II.—RECTANGULAR RESOLUTION OF FORCES. Resolved part of a force. Magnitude of the resultant of two forces including any angle. Lami's Theorem. Principle of Work for a particle .. 24—42

Examination Paper I. 43

CHAPTER III. — EASY PROBLEMS ON EQUILIBRIUM. Types of Forces. The Statical Hypotheses. The Inclined Plane. The solution of Problems by Force Triangles, by Resolution, and by Lami's Theorem 44—66

CHAPTER IV.—THE TRANSMISSION OF FORCE. Rigid Bodies. Equilibrium of three forces. Heavy Bodies .. .. 67—85

Examination Paper II. 86

CHAPTER V.—MOMENTS OF FORCES IN ONE PLANE. Definition, algebraic sign, and geometrical representation of Moments. Varignon's Theorem .. .. 87—103

CHAPTER VI.—PARALLEL FORCES: their resultant when like and when unlike. Conditions of equilibrium of three parallel forces. Experimental verifications .. 104—118

CHAPTER VII.—SYSTEMS OF PARALLEL FORCES. Equilibrium of a loaded beam or heavy lever. Resultant and centre of parallel forces at given points in a straight line. COUPLES; their moments and properties. Resultant of two or more Couples .. 119—133

Examination Paper III. 134

CHAPTER VIII.—MACHINES; their mechanical advantage. The Lever; the three classes of lever. The Wheel and Axle; The Windlass and Capstan. PRINCIPLE OF WORK FOR MACHINES .. .. 135—159

CHAPTER IX.—THE PULLEY AND SCREW. The three Systems of Pulleys. Work as applied to Pulleys. The Screw. The Differential Pulley and Screw .. 160—182

Examination Paper IV. 183

PAGES

CHAPTER X.—CENTRES OF PARALLEL FORCES. DEFINITION OF THE CENTRE OF GRAVITY. Every heavy body has one and only one C.G. Mass-Centre defined. C.G. of a Straight Line, Circle, or Sphere. C.G. of a Parallelogram .. .. 184—197

CHAPTER XI.—PROPERTIES OF THE CENTRE OF GRAVITY. Equilibrium of a Heavy Body about a Fixed Point. To find the C.G. of a Lamina. Equilibrium of a Body resting on its Base. Stable, Unstable, and Neutral Equilibrium .. .. 198—212

CHAPTER XII.—DETERMINATION OF THE CENTRE OF GRAVITY. Triangular Lamina. Triangular Wire. Polygon. C.G. of a portion of a body, and of a system of weights in one plane. Work done in raising C.G. of weights. C.G. of a Triangular Pyramid or Cone .. .. 213—232

Examination Paper V. 232

CHAPTER XIII.—BALANCES. Requisites of a good Balance. False Balances. The Common or Roman Steelyard. The Danish Steelyard .. .. .. 233—245

Examination Paper VI. 246

CHAPTER XIV.—CONDITIONS OF EQUILIBRIUM OF FORCES IN ONE PLANE. Necessary and Sufficient Conditions. Properties of Couples .. .. .. 247—257

CHAPTER XV.—PROBLEMS ON COPLANAR FORCES. Rigid Bodies. Frameworks. Graphic Methods. Bow's Notation .. .. 258—280

CHAPTER XVI.—THE PRINCIPLES OF VIRTUAL WORK AND VIRTUAL VELOCITIES .. .. 281—294

Examination Paper VII. 295

CHAPTER XVII.—THE LAWS OF FRICTION. Equilibrium on Rough Inclined Plane, and in Rough Screw-Press 296—311

CHAPTER XVIII.—FURTHER DETERMINATIONS OF CENTRE OF GRAVITY. Frustum of Cone, Circular Arc and Sector, Spherical Zone and Sector .. .. 312—328

Examination Paper VIII. 329

MISCELLANEOUS EXERCISES .. .. 330—346

ANSWERS .. .. 347—357

EXAMINATION QUESTIONS .. .. 358—366

SUMMARIES OF RESULTS AND EXAMPLES at the end of each Chapter.

STATICS.

PART I.

EQUILIBRIUM OF FORCES AT A POINT.

CHAPTER I.

THE PARALLELOGRAM OF FORCES.

1. **Statics** is that branch of Mechanics which deals with forces applied to a body or a number of bodies at **rest.**

For the present we shall only consider the properties of forces applied to a single **particle.** In Chap. IV. we shall treat of forces acting on a body of extended size.

2. Force is defined as that which changes or tends to change a body's state of rest or motion.

A **particle** is a body whose size is so small that it may be regarded as a quantity of matter collected at a single point.

3. Two forces are said to be **equal** if, when they are applied to the *same* body for *equal* intervals of time, they tend to impart the same velocity, or change of velocity, to the body.

If one force imparts double the change of velocity that would be imparted on the same body in an equal interval of time by another force, the first force is said to be double the second, and so on. Thus forces are proportional to the velocities they would impart to the same body in equal intervals of time. In this way different forces may be compared, and the **magnitude** of a force may be measured in terms of any force which is chosen as the standard or "unit of force."

The **direction** of a force is defined as the direction of the velocity which it tends to impart to the body on which it acts.

Thus force is characterised by having both **magnitude** and **direction.**

4. When a force is applied to start a body from **rest,** the *actual distance* traversed in any time is proportional to the force*, and the *actual direction of motion* is the direction of the force. Thus, if a body initially *at rest* is pulled with a force of 2 lb., it will move in the direction in which it is pulled, and the distance which it will move over in one minute will be twice what it would move over if pulled by a force of 1 lb. for the same time.

5. Equilibrium.—Resultant Force.—If two equal forces act on the same particle in exactly opposite directions, the motion which one force tends to impart is exactly the reverse of that which the other force tends to impart. Hence, if the particle be at rest it will remain at rest, and if it be in motion its motion will be unaltered, that is to say it will continue to move uniformly in a straight line. The forces are said to *balance,* or to be in *equilibrium.*

If several independent forces are acting simultaneously on a single particle, the particle's state of rest or motion may either remain constant or alter. If it remain constant, the forces are said to **balance,** or to be in **equilibrium.** It it alter, this alteration could be produced by a certain single force of suitable magnitude applied in a suitable direction. This force is called the **resultant** of the several forces.

Conversely, any forces which have a given force for their resultant are called **components** of the given force.

It is important to remember that forces are in equilibrium when the body on which they act moves uniformly in a straight line, just as much as when it is at rest. For it may be inferred from Newton's First Law of Motion that if a body is moving uniformly in a straight line either (i) there are no forces acting on it, or (ii) the forces acting (since they produce no change of motion) must neutralise each other, *i.e.* must be in equilibrium.

* This follows from the equations of dynamics $P = mf$, $s = \frac{1}{2}ft^2$.

6. **Systems of forces.**—The forces with which we shall deal in *Statics* will always be supposed to be kept in equilibrium. But it is often necessary to consider the properties of some of the forces apart from the rest. Any number of forces may be called a **system** of forces, and may be subsequently referred to as "the system" to avoid repeatedly specifying what forces are meant.

When a number of forces have a resultant we may reduce them to a system in equilibrium by applying an additional force equal and opposite to that resultant. For the whole system is then equivalent to two equal and opposite forces (viz., the original resultant and the added force), and is therefore in equilibrium. This additional force is sometimes called the **equilibrant.**

Conversely, when any number of forces are in equilibrium, each force is equal and opposite to the resultant of all the rest taken together.

7. **Point of application of a force.**—The conditions of equilibrium of a system of forces do not depend on the nature and *mass* of the body on which they act, but only on the forces themselves. We may, therefore, speak of a force as **acting at a point,** meaning a force "applied to any particle placed at that point." And the **point of application** of the force is "the point at which the particle acted on by the force is situated."

These must, however, be regarded merely as abbreviations, for force can only act where there is matter for it to act on.

8. **Statical units of force.**—The forces which occur most frequently in Statics are those due to *weight.* Hence the most convenient statical unit of force is the *weight of a pound,* and this we call "a force of 1 lb." Larger forces may, however, be measured in hundredweights or tons.

If the French system of weights and measures is used, the statical unit of force will be the weight of a gramme or of a kilogramme (1,000 grammes), according to which is the most convenient.

Thus, in Statics, forces are always measured in gravitation units unless otherwise stated.

9. **Forces may be represented by straight lines.**—In order to define a force statically, it is necessary to specify

(i.) its point of application,
(ii.) its direction,
(iii) its magnitude.

All these data will be specified by a straight line of finite length, provided that

(i.) the line is drawn from the *point of application* of the force,
(ii.) it is drawn pointing in the *direction* of the force,
(iii.) its *length* is proportional to the *magnitude* of the force.

Such a straight line is said to *represent* the force.

The *sense* of the direction may be shown by an arrow drawn on or by the side of the line, or by the *order* of the letters used in naming the line. Thus **AB** represents a force acting from **A** towards **B**, **BA** a force acting from **B** towards **A**.

10. **On the choice of a scale of representation.**—In order that the length of a straight line may represent the magnitude of a force, the line should properly contain as many units of length as the force contains units of force. Thus, if a line 1 inch long represents a force of 1 lb., a line 2 inches long will represent a force of 2 lbs., and so on. Very often, however, it is necessary to adopt some other scale of representation suggested by the conditions of the problem. *We may so choose the scale of representation that* one *of the forces is represented by a straight line of any length we please.* When this has been done, a line of double that length will represent double that force, and so on, so that the lines representing all *other* forces will be fully determined.

[Thus it might be convenient for some reason to agree to represent a force of 7 lb. by a length of $\frac{1}{4}$ inch. On this scale a length of $\frac{1}{2}$ inch would represent a force of 14 lb., and so on.]

11. It is often necessary to represent a force *in magnitude and direction only* by a straight line *not* drawn from its point of application. A force will be represented to this extent by *any straight line drawn equal and parallel* to the line which fully represents it, for *parallel straight lines are to be regarded as having the same direction.*

12. **THE PARALLELOGRAM OF FORCES.**—**If two forces acting on the same particle be represented by two adjacent sides of a parallelogram, drawn from their point of application, their resultant shall be represented by the diagonal of the parallelogram drawn from that point.**

Let two constant forces P, Q be applied to a particle in directions **AB, AD** respectively (Fig. 1).

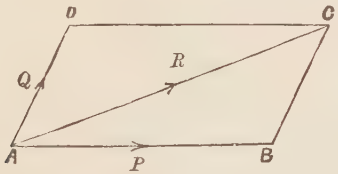


Fig. 1.

Let **AB** represent the velocity which would be produced by the force P acting on the particle for time t; and let **AD** represent the velocity which would be produced by the force Q acting on the particle for time t.

Then by the principle of the Physical Independence of Forces (*Tutorial Dynamics*, § 174), the change of velocity produced by the forces P and Q conjointly in a given time t, will be the resultant of the changes of velocity which would be produced by P and Q acting separately, each for the given time t.

Thus if we complete the parallelogram **ABCD**, the diagonal **AC** will represent the resultant of the two velocities **AB** and **AD** (*Tutorial Dynamics*, § 161), and will therefore represent the velocity produced in time t by the joint action of P and Q.

Hence the single force which will produce the velocity **AC** in time t, will be the resultant of the forces P and Q.

Hence (i) the resultant acts in the direction **AC**, and therefore **AB**, **AD**, and **AC** respectively represent P, Q, and the resultant *in direction*

Also (ii) since the velocities produced in the particle in the time t are proportional to the forces which produce them (§ 3), therefore P, Q, and the resultant are proportional to **AB**, **AD**, and **AC**.

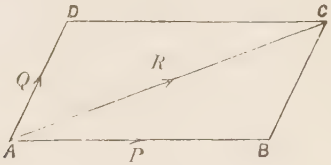


Fig. 1.

If, then, we choose the scale of representation of the forces so that the line **AB** represents the force P in magnitude, then the lines **AD** and **AC** will respectively represent Q and the resultant *in magnitude.*

Thus if two forces P, Q are represented in magnitude and direction by the sides **AB, AD** *of the parallelogram* **ABCD,** *then their resultant will be represented in magnitude and direction by the diagonal* **AC.**

It is very important to notice that this proof depends entirely on the principle of the Physical Independence of Forces, while the truth of this principle is deduced ultimately from experiment. A theoretical proof of the parallelogram of forces can be given, which is independent of this principle ; but it is very long and intricate, and in some respects unsound as a logical argument.

13. Experimental verification of the Parallelogram of Forces.

(*a*) Mechanical Details.—Take three strings ; knot them together in a point. To their ends attach any three weights **P, Q, R**, say P, Q, R lb., respectively (any two of which are together greater than the third). Allow one string to hang freely with its suspended weight **R,** and pass the other two over two smooth pulleys **H, K,** fixed in front of a vertical wall (Fig. 2).

(*b*) Geometrical Construction.—When the strings have taken up a position of equilibrium with the knot at

A, measure off along **AH**, **AK** lengths **AB**, **AD**, containing P and Q units of length, respectively. On the wall complete the parallelogram **ABCD**, and join **AC**.

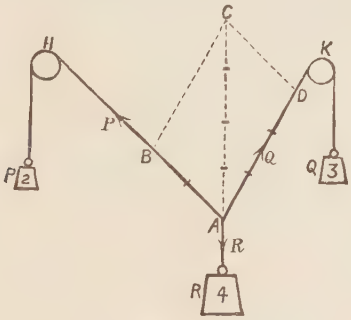


Fig. 2.

(*c*) Observed Facts.—*Then it will be invariably found*

(i.) *that the diagonal* **AC** *is vertical,*

(ii.) *that* **AC** *contains* R *units of length.*

(*d*) Deductions.—Now the knot **A** is in equilibrium under the pulls P, Q, R acting along the strings, respectively. Therefore the resultant of P, Q is equal and opposite to the weight R.

Therefore it is a force of R lb. acting vertically upwards.

But **AC** is vertical,

$\therefore$ **AC** represents the resultant in direction.

Also **AC** contains R units of length ;

$\therefore$ **AC** represents the resultant in magnitude.

But **AB, AD** represent the forces P, Q.

Therefore the diagonal of the parallelogram represents the resultant of the two forces which are represented separately by its sides. [Q.E.D.]

14. Fig. 2 is drawn for the case in which $P = 2$ lb., $Q = 3$ lb., $R = 4$ lb. The measured lengths **AB, AD** must therefore contain 2 and 3 units of length respectively. When the parallelogram is constructed, the diagonal **AC** will be found to be vertical and to contain 4 units of length. Each experiment proves the Parallelogram of Forces to hold good for one particular set of forces only. To give a satisfactory proof it would be necessary to perform a large number of such experiments, using different arrangements of weights.

Example.—To find graphically the resultant of forces of 7 lb. and 11 lb., whose directions include an angle of 60°.

Take any unit of length and measure off **AB, AD** containing 7 and 11 units respectively, making $\angle$ **BAD** = 60°. Complete the parallelogram **ABCD.**

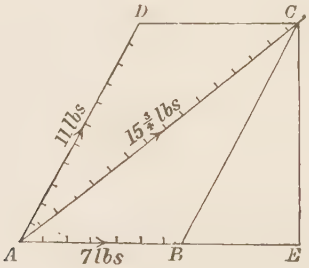


Fig. 3.

Then **AC** represents the resultant.

On **AC** mark off from **A** a scale of the selected units. Then **C** will be found to lie between the 15th and 16th marks, so that **AC** contains about $15\frac{3}{4}$ units.

Therefore the resultant force = $15\frac{3}{4}$ lb. wt. roughly.

15. **Half the parallelogram is sufficient.** Since **BC** is equal and parallel to **AD,** it represents the force Q in magnitude and direction (though not in position—see § 11). Hence, if two forces acting on a particle are represented in magnitude and direction by two sides of a triangle, **AB, BC,** their resultant is represented in magnitude and direction by the third side **AC,** its point of application being that of the forces. This leads to the following theorem.

16. The Triangle of Forces.—*If three forces acting on the same particle can be represented in magnitude and direction (but not in position) by the sides of a triangle taken in order,* they shall be in equilibrium.*

Let three forces P, Q, R, acting on the same particle at **O**, be represented in magnitude and direction (but not in

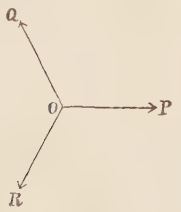


Fig. 4.

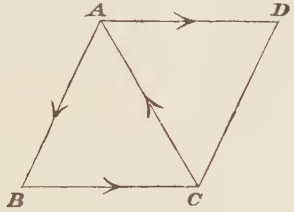


Fig. 5.

position) by the sides **BC**, **CA**, **AB** of the triangle **ABC**, respectively.

Then shall P, Q, R be in equilibrium.

Complete the parallelogram **ABCD.**

The forces R, P are represented in magnitude and direction by **AB**, **AD**, and they act at **O.** Hence by the Parallelogram of Forces, their resultant is similarly represented by **AC**, and also acts at **O.**

But Q is represented by **CA.**

Therefore the resultant of R, P is equal and opposite to the third force Q.

Therefore the three forces are in equilibrium. [Q.E.D.]

The angle between any two of the forces is the supplement of the corresponding interior angle of the triangle.

Thus

$\angle$ between R, P (*i.e.*, $\angle ROP$ in Fig. 4) $= \angle$ **BAD** $= 180° - \angle$ **ABC.**

OBSERVATION.—The three forces *must not act along the sides of the triangle.* They must act *at a point* in directions parallel to these sides, as in Fig. 4; otherwise they cannot be applied to the same particle, and the proof fails.

* The sides of a triangle or polygon are said to be *taken in order* when of any two adjacent sides one is drawn towards and the other away from their common angular point.

17. **Converse of the Triangle of Forces.**

If three forces acting on a particle are in equilibrium, any triangle whose sides are parallel to the directions of the forces shall have the lengths of these sides proportional to the magnitudes of the forces.

Let P, Q, R be three forces in equilibrium acting at **O**, and let **ABC** be any triangle whose sides **BC, CA, AB** are parallel to P, Q, R.

Then, if the scale of representation be properly chosen, **BC, CA, AB** shall represent P, Q, R in magnitude as well as direction.

For let the length **BC** be chosen to represent P. (§ 10.)

If **CA** does not represent Q, let **CK** represent Q.

Then, by the Parallelogram or Triangle of Forces, the resultant of P and Q is represented by **BK**. But P, Q, R

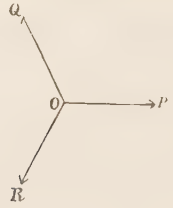


Fig. 6.

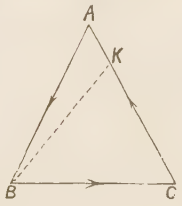


Fig. 7.

are in equilibrium. Hence R must be represented by **KB**, equal and opposite to **BK**. But this is contrary to hypothesis, since R acts in the direction **AB**. Therefore Q cannot be represented in magnitude by any other length than **CA**, and therefore also, by the above reasoning, R is represented by **AB**.

Observation.—The above conditions may be expressed by the relations $\frac{P}{Q} = \frac{\mathbf{BC}}{\mathbf{CA}}$, $\frac{Q}{R} = \frac{\mathbf{CA}}{\mathbf{AB}}$, $\frac{R}{P} = \frac{\mathbf{AB}}{\mathbf{BC}}$, which may also be written

$$\frac{P}{\mathbf{BC}} = \frac{Q}{\mathbf{CA}} = \frac{R}{\mathbf{AB}}.$$

It is proved in Euclid, Book VI., Prop. 4, that any triangle whose angles are equal to those of **ABC** has its sides proportional to those of **ABC**. Hence, if *any* triangle be drawn whose sides are parallel to **BC, CA, AB**, these sides also will represent P, Q, R, but on a different scale.

18. The Polygon of Forces.—*If any number of forces acting on the same particle can be represented in magnitude and direction by the sides of a closed polygon taken in order, they shall be in equilibrium.*

Let the forces P, Q, R, S, acting on a particle at **O**, be represented in magnitude and direction (but not in

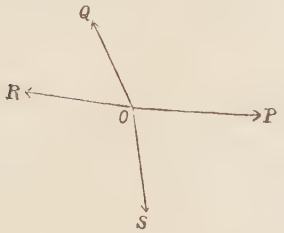


Fig. 8.

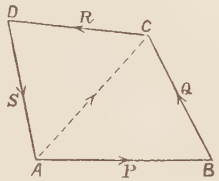


Fig. 9.

position) by the sides **AB, BC, CD, DA** of a polygon **ABCD.**

Then shall the forces be in equilibrium.

For, as in the Triangle of Forces, or § 15, the resultant of the forces P, Q is represented in magnitude and direction by **AC.**

Therefore the resultant of P, Q, R is the same in magnitude and direction as that of forces **AC** and **CD,** and therefore similarly represented by **AD.**

But the force S is represented by **DA.**

Therefore S is equal and opposite to the resultant of P, Q, R.

Therefore the forces P, Q, R, S are in equilibrium.

[Q.E.D.]

The observations at the end of § 16 are equally applicable to the Polygon of Forces.

We have considered the case of four forces, but the proof may be similarly extended to any number of forces.

19. **To construct the resultant of any number of forces acting on a particle.**

Let the given forces be represented by the straight lines **AB, BC, CD,** taken in order, forming all the sides but one of a polygon. Then, if the polygon be completed by draw-

ing the remaining side *from* **A**, the extremity of the first side, *to* **D**, the extremity of the last side, the line **AD** will represent the resultant force.

This is evident from the last article.

It is immaterial in what order the forces are represented. The form of the polygon will depend on which force is represented first, which next, and so on; but the line representing the resultant will be the same in every case.

[For, consider the case of two forces P, Q, acting at **A** (Fig. 1). If we represent P first and Q second, the lines representing them will be **AB, BC,** respectively, and the resultant will be represented by **AC.** If we represent Q first and P second, the lines representing them will be the opposite sides **AD, DC,** respectively, and therefore the resultant will still be represented by **AC.** The same property may be extended to any number of forces by interchanging their order of successive representation, taking two at a time.]

20. **Converse of the Polygon of Forces.**

If any number of forces acting on a particle are in equilibrium, a closed polygon can be drawn whose sides represent these forces both in magnitude and direction.

Let the forces P, Q, R, S, acting at **O**, be represented by **AB, BC, CD, DE,** respectively, these lines being placed end

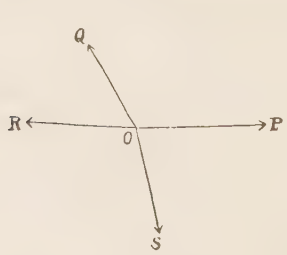


Fig. 10.

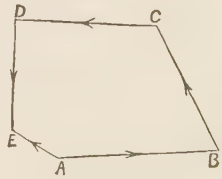


Fig. 11.

to end. Then, if the figure **ABCDE** be not a closed polygon, the forces will have a resultant represented by **AE** (§ 19), and will, therefore, not be in equilibrium.

Therefore the lines representing forces in equilibrium must be capable of forming a closed polygon.

21. *The converse of the Polygon of Forces is less complete than that of the Triangle.*—

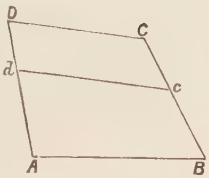


Fig. 12.

For, if there are more than three forces, we can draw any number of different polygons, such as **ABCD, ABcd** (Fig. 12), each having its sides parallel to the directions of the forces, and the sides of one polygon are not necessarily proportional to those of another. The sides of each polygon represent *a* system of forces in equilibrium acting in these directions, but not necessarily *the* system we started with. In fact there are any number of such systems all different. We cannot, therefore, except in the case of three forces, determine the ratios of the forces from merely knowing their directions.

22. **Applications of the Parallelogram, Triangle, and Polygon of Forces.**

(1) *The resultant of two equal forces bisects the angle between them.*

(2) *If any three forces are in equilibrium, any two of the forces are together not less than the third.* For, in the Triangle of Forces, any two sides are together greater than the third (Euc. I. 20).

If two forces are together *equal* to the third, they will balance if the first two forces act in the same straight line and in the opposite sense to the third.

(3) *The resultant of two forces P, Q is greatest* when both act in the same direction, and is then $P + Q$. It is *least* when they act in opposite senses in the same straight line, and is then either $P - Q$ acting in the direction of P or $Q - P$ in the direction of Q.

This is an easy inference from (2).

(4) *If three equal forces are in equilibrium, the angle between any two of them is* 120°. For the Triangle of Forces is equilateral; therefore each of its angles is 60°, and the angle between the corresponding pair of forces

$$= 180° - 60° = 120°.$$

(5) *The Perpendicular Triangle of Forces.*—If three forces proportional to the sides of a triangle act on a particle in directions perpendicular to these sides taken in order, they shall be in equilibrium.

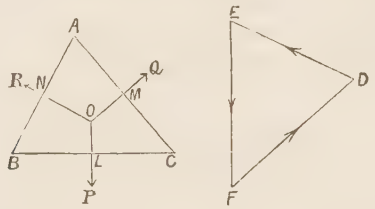


Fig. 13.

For if the triangle **ABC** be turned through a right angle into the position **DEF**, its sides, taken in order, will be brought parallel to the forces, and will therefore represent the forces both in magnitude and direction. Therefore they will be in equilibrium.

Conversely, if three forces acting perpendicularly to the sides of a triangle keep a particle in equilibrium, they shall be proportional to these sides.*

(6) *The Perpendicular Polygon.*—Similarly, if any number of forces proportional to the sides of a closed polygon act on a particle in directions perpendicular to these sides taken in order, they shall be in equilibrium.

(7) *If two forces be represented by the sides of a triangle* ***both drawn from*** *their point of application, their resultant is represented by twice the bisector of the base drawn from that point.*

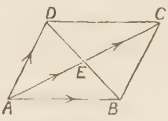


Fig. 14.

For let **AB, AD** represent the forces. Complete the parallelogram of forces **ABCD**; then **AC** represents the resultant. But the diagonals of a parallelogram bisect each other. Hence **AC** bisects **BD** in **E**, and **AC** is twice the bisector **AE**.

* This theorem has an important application to Hydrostatics, in that the pressure at a point in a fluid is the same in all directions.

(8) *To find where a particle* **O** *must be placed inside a triangle* **ABC** *so that it may be in equilibrium under forces to the vertices represented by* **OA, OB, OC.**

Let **D** be the middle point of **BC**. Then, by the last proposition, the forces **OB, OC** have a resultant **2OD** along **OD**. This must be equal and opposite to the third force **OA**. Hence **O** must lie on the bisector **AD** at a point such that $OA = 2DO$.

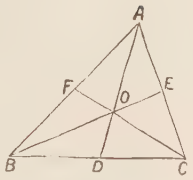


Fig. 15.

$$\therefore \quad DA = 3DO \quad \text{and} \quad DO = \tfrac{1}{3}DA.$$

[The point **O** is the *centre of gravity* of the triangle ; see Chap. XII.]

23. **Two forces act along two sides of a triangle, and their magnitudes are given multiples of these sides. To find their resultant.**

Case 1. Let the forces be m . **AB** acting along **AB**, and n . **AC** along **AC**, (Fig. 16). Divide **BC** at **O** in the ratio $n : m$, (*i.e.* so that $BO : OC = n : m$). Then the required resultant will be represented by $(m + n)$. **AO**.

For (by § 15) the force m . **AB** is equivalent to forces

$\left\{\begin{array}{l} m \text{ . } \mathbf{AO} \text{ along } \mathbf{AO}, \\ m \text{ . } \mathbf{OB} \text{ at } \mathbf{A} \text{ parallel to } \mathbf{OB}. \end{array}\right.$

Similarly the force n . **AC** is equivalent to forces

$\left\{\begin{array}{l} n \text{ . } \mathbf{AO} \text{ along } \mathbf{AO}, \\ n \text{ . } \mathbf{OC} \text{ at } \mathbf{A} \text{ parallel to } \mathbf{OC}. \end{array}\right.$

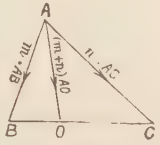


Fig. 16.

Hence the two forces together are equivalent to

$\left\{\begin{array}{l} (m+n) \text{ } \mathbf{AO} \text{ along } \mathbf{AO}, \\ n \text{ . } \mathbf{OC} - m \text{ . } \mathbf{BO} \text{ at } \mathbf{A} \text{ parallel to } \mathbf{BC}. \end{array}\right.$

But $BO : OC = n : m$, (by construction) ;

i.e. $\quad n \text{ . } OC = m \text{ . } BO, \quad \therefore n \text{ . } OC - m \text{ . } BO = 0.$

Thus the component parallel to **BC** vanishes, and therefore the two forces together are equivalent to

$$(m + n)\ \mathbf{AO} \text{ along } \mathbf{AO}.$$

This is conveniently represented by the notation

$$\left.\begin{matrix} m\ .\ \mathbf{AB} \\ n\ .\ \mathbf{AC} \end{matrix}\right\} = (m + n)\ .\ \mathbf{AO}.$$

To find the point **O** we have $\mathbf{BO} : \mathbf{OC} = n : m$;

$$\therefore \mathbf{BO} : \mathbf{BC} = n : n + m;\ \textit{i.e.}\ \mathbf{BO} = \frac{n}{n + m}\,\mathbf{BC}.$$

Case 2. Let the forces be m . **AB** acting along **AB**, and n . **CA** along **CA** (in the opposite sense to **AC**, Fig. 17).

If $m = n$, the resultant force is m . **CB** parallel to **CB**, (§ 15).

If $n < m$, divide **BC** externally in the ratio $n : m$; (*i.e.* find a point **O** in **CB** *produced* such that

$$\mathbf{OB} : \mathbf{OC} = n : m).$$

Then the required resultant will be represented by

$$(m-n)\mathbf{AO}.$$

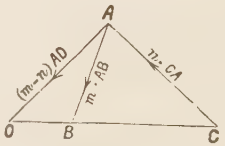


Fig. 17.

For (by § 15),

$$m\ .\ \mathbf{AB} = \begin{cases} m\ .\ \mathbf{AO} \text{ along } \mathbf{AO}, \\ m\ .\ \mathbf{OB} \text{ at } \mathbf{A}, \text{ parallel to } \mathbf{OB}. \end{cases}$$

$$n\ .\ \mathbf{CA} = \begin{cases} n\ .\ \mathbf{CO} \text{ at } \mathbf{A}, \text{ parallel to } \mathbf{CO}, \\ n\ .\ \mathbf{OA} \text{ along } \mathbf{OA}. \end{cases}$$

Thus

$$\left.\begin{matrix} m\ .\ \mathbf{AB} \\ n\ .\ \mathbf{CA} \end{matrix}\right\} = \begin{cases} (m-n)\ \mathbf{AO} \text{ along } \mathbf{AO}, \\ (m\ .\ \mathbf{OB} - n\ .\ \mathbf{OC}) \text{ at } \mathbf{A}, \text{ parallel to } \mathbf{BC}. \end{cases}$$

But m . **OB** = n . **OC**, (by construction) :

$$\therefore m\ .\ \mathbf{OB} - n\ .\ \mathbf{OC} = 0.$$

Thus

$$\left.\begin{matrix} m\ .\ \mathbf{AB} \\ n\ .\ \mathbf{CA} \end{matrix}\right\} = (m-n)\ .\ \mathbf{AO}.$$

To find the point **O** we have $\mathbf{BO} : \mathbf{OC} = n : m$.

$$\therefore \mathbf{BO} : \mathbf{BC} = n : m-n;\ \textit{i.e.}\ \mathbf{BO} = \frac{n}{m - n}\,\mathbf{BC}.$$

If $m < n$ a similar construction holds, we again divide **BC** externally in the ratio $n : m$; but **O** will lie to the right of **C**; and the resultant will be $(m - n)$ **AO**, *i.e.* $(n - m)$ **OA**.

24. To find the magnitude of the resultant of two forces P, Q in directions at right angles to one another.

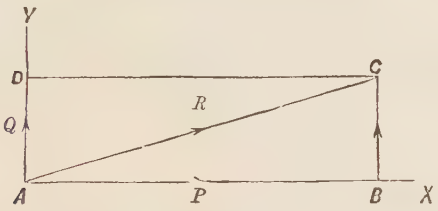


Fig. 18.

Let **AB, AD** represent two forces P, Q.
Complete the rectangle **ABCD.**
Then **AC** represents the resultant force.
Let this be R. Then, by Euclid I. 47,

$$\mathbf{AC}^2 = \mathbf{AB}^2 + \mathbf{BC}^2 = \mathbf{AB}^2 + \mathbf{AD}^2;$$

$$\therefore \quad R^2 = P^2 + Q^2 \ldots\ldots\ldots\ldots\ldots\ldots (1);$$

or **resultant force** $R = \sqrt{(P^2 + Q^2)}$,

Again, to find the direction of the resultant, we must calculate the magnitude of the angle **CAB.** Since $\angle$ **CBA** is a right angle,

$$\tan \mathbf{CAB} = \mathbf{CB}/\mathbf{AB} = \mathbf{AD}/\mathbf{AB},$$

i.e. $$\tan \mathbf{CAB} = Q/P \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

This equation determines tan **CAB,** and the magnitude of the angle can then be determined by using Trigonometrical Tables, in which are registered the tangents of all angles between 0° and 90°.

25. **Scalars and Vectors.**—Since forces on a particle are compounded by the same law as velocities and accelerations, viz. by the Triangle Law or Parallelogram Law, it

follows that all the properties proved, both in this and the next chapter, hold equally for velocities or accelerations.

Besides forces, velocities, and accelerations, other quantities occur in Mechanics which possess magnitude and direction, and which are combined by the Triangle or Parallelogram Law. To all such quantities we apply the general name of **vector**; while to all quantities which are combined in the ordinary way, viz. by addition or subtraction (such as money, time, volume, etc.), we apply the general term of **scalar.** Thus mass is a scalar quantity, because if two masses are combined the resulting mass is the sum of the original masses.

26. Typical Problems.—

Examples.—(1) In Fig. 2 (§ 13), find graphically the inclinations of the strings **AH** and **AK** to the vertical; if P, Q, and R are masses of 5, 6, and 7 lb. respectively.

Draw **AC** vertical and equal to 7 units of length; (a convenient unit of length will be a quarter of an inch). By Euc. I. 22 complete the triangle **ABC,** making **AB** = 5 units, **BC** = 6 units. Complete the parallelogram **ABCD,** and sketch in the rest of the figure. Then **AD** = **BC** = 6 units; thus the parallelogram **ABCD** satisfies the theoretical requirements. The actual lengths of the strings **AH, AK,** etc., do not affect the problem. By means of a protractor we can now measure the angles **CAB, CAD**; we shall find them about 57° and $44\frac{1}{2}$° respectively.

(2) Solve the preceding problem by calculation, instead of by accurate drawing.

Sketch Fig. 2, assuming **AB, AD, AC** respectively equal to 5, 6, 7 units of length; then **BC** = **AD** = 6. Thus in the triangle **ABC,** using the usual trigonometrical notation, a = **BC** = 6, b = **CA** = 7, c = **AB** = 5. Thus by a well-known formula in **Trigonometry,**

$$\cos \mathbf{BAC} = \frac{b^2 + c^2 - a^2}{2bc} = \frac{49 + 25 - 36}{70} = \cdot 5428;$$

whence, using the Table of Cosines, we see that **BAC** = $57°7\frac{1}{2}'$.

Again $\cos \mathbf{DAC} = \cos \mathbf{ACB} = \frac{a^2+b^2-c^2}{2ab} = \frac{36+49-25}{84} = \cdot 7143;$

whence **DAC** = 44°25′.

(3) To find the resultant of three forces of 1 lb. acting at a point, the angle between the first and second being 90°, and that between the second and third being 45°.

Fig. 19 represents the three forces acting at the point **O**. Let **AB**, **BC**, **CD** represent these forces in magnitude and direction (Fig. 20) ; then **AD** represents the resultant in magnitude and direction (§ 19).

Now by § 16, $\angle ABC = 180° - \angle POQ = 90°$;

$\angle BCD = 180° - \angle QOR = 135°$.

Also since $\triangle ABC$ is right-angled and isosceles,

$\angle BCA = 45°$;

hence $\angle ACD = \angle BCD - \angle BCA = 135° - 45° = 90°$.

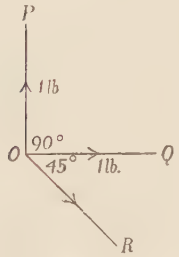


Fig. 19.

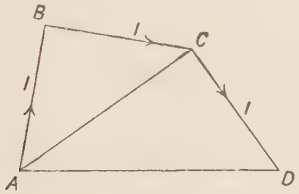


Fig. 20.

Then, by Euc. I. 47, $AC^2 = AB^2 + BC^2 = 1^2 + 1^2 = 2$;

$\therefore AD^2 = AC^2 + CD^2 = 2 + 1^2 = 3$.

Thus $AD = \sqrt{3} = 1{\cdot}732$ lb.

Thus the resultant is a force of 1·732 lb. acting at **O** in a direction parallel to **AD**. Measuring the angle **BAD**, we find that it is about 80° ; hence the resultant is inclined to **OP** at an angle of 80°.

Summary of Results.

A straight line can represent a force (i.) in point of application, (ii.) in direction and sense, (iii.) in magnitude. (§ 9.)

The Parallelogram of Forces.—If two forces acting on the same particle be represented by two adjacent sides of a parallelogram drawn from their point of application, their resultant shall be represented by the diagonal of the parallelogram drawn from that point. (§ 12.)

The Triangle and Polygon of Forces.—If three or more

forces acting on the same particle can be represented in magnitude and direction (but not in position) by the sides of a triangle or closed polygon taken in order, they shall be in equilibrium. (§§ 16, 18.)

Converse of "Triangle."—If three forces (P, Q, R) acting on a particle balance, *any triangle* (**ABC**), whose sides are parallel to the forces, shall have these sides proportional to their magnitudes (§17), or

$$\frac{P}{\mathbf{BC}} = \frac{Q}{\mathbf{CA}} = \frac{R}{\mathbf{AB}}$$

Converse of "Polygon."—If *more* than three forces balance, *a polygon can be drawn* whose sides represent them in magnitude and direction. (§§ 20, 21.)

Resultant of forces m . **AB** and n . **AC** along **AB, AC** is force $(m+n)$ **AO** along **AO,** where m . **BO** = n . **OC.** (§ 23.)

Resultant of two perpendicular forces, P, Q is R, where

$$R^2 = P^2 + Q^2 \ldots\ldots\ldots (1). \quad (\S\ 24.)$$

Also the angle between R and P is given by the equation

$$\tan\theta = \frac{Q}{P} \quad \ldots\ldots\ldots (2). \quad (\S\ 24.)$$

EXAMPLES I.

(*Selected Problems* :—1, 5, 7, 8, 9, 12, 15, 20, 21, 23, 25, 28, 29, 31, 32.)

1. Draw a diagram, as well as you can to scale, showing the resultant of two forces, equal to the weights of 6 and 12 lb., acting on a particle, with an angle of 60° between them ; and, by measuring the resultant, find its numerical value.

2. The greatest resultant which three given forces acting at a point can have is 30 lb., and the least is 2 lb. What is the magnitude of the greatest force ?

3. The greatest resultant which three given forces acting at a point can have is 30 lb., and the least is 0. Find the limits between which the greatest force must lie.

4. **ABC** is a triangle. Find the resultant of forces at **A** represented in magnitude and direction and sense by

(i.) 3**AB** and 5**AC** ; (iii.) **BA** and 3**AC** ; (v.) 4**AB**, 5**AC**, 6**BC** ;
(ii.) 2**BA** and 3**CA** ; (iv.) 4**AB** and 5**CA** ; (vi.) 3**AB**, 4**CA**, 5**CB**.

5. A body is pulled north, south, east, and west by four strings whose directions meet in a point, and the forces of tension in the strings are equal to 10, 15, 20, and 32 lb. weight respectively. What is the magnitude of the resultant?

6. Forces P, $2P$, $3P$ act at a point in directions parallel to the sides of an equilateral triangle taken in order. Find their resultant.

7. Show how to find, graphically or otherwise, the resultant of a number of forces acting on a rigid body at one point, and apply your method to find the resultant of forces 1 in an easterly direction, $\sqrt{2}$ in a north-easterly direction, and 1 to the north.

8. What angle must two forces of 5 and 12 lb. include if they are balanced by a force of 13 lb.?

9. Show that the resultant of two perpendicular forces $P + Q$ and $P - Q$ is equal in magnitude to the resultant of two perpendicular forces $\sqrt{2}P$ and $\sqrt{2}Q$.

10. The sides **BC**, **CA**, **AB** of a triangle are bisected in **D**, **E**, **F**, respectively. Find the resultant of forces represented by **DA**, **EB**, **FC**.

11. **ABC** is any triangle. At **A** are applied forces k.**BC**, l.**CA**, m.**AB**, parallel to **BC** and along **CA**, **AB**, respectively. Show that their resultant is the resultant of forces $(k - l)$ **AC** along **AC** and $(m - k)$ **AB** along **AB**, and hence find where it cuts the base **BC**.

12. **ABDC** is a parallelogram; **E** is a point in **AC**. Find a point **F** in **BD** such that the resultant of forces represented by **AE** and **AF** may act in the direction **AD**.

13. If **A**, **B**, **C** are any three points in a straight line, **O** any point not in that straight line; if a force represented in magnitude and direction by **OA** act from **O** to **A**; if a force represented in magnitude and direction by **BO** act from **B** to **O**, and a force represented in magnitude and direction by **CO** act from **C** to **O**; then show that the resultant of these three forces cuts the straight line **ABC** in point **D** such that **AB** = **CD**.

14. Forces P, Q, whose resultant is R, act at a point **O**, and a line is drawn meeting their directions in **L**, **M**, **N**; show that

$$P \quad P/\mathbf{OL} + Q/\mathbf{OM} = R/\mathbf{ON}.$$

15. Find by a graphical construction the angle at which two forces, each equal to 50 lb. weight, must act at a point, that they may have a resultant of 75 lb. weight.

16. If the resultant of forces represented by lines drawn from a point **P**, to the angular points of a quadrilateral, be of constant magnitude, prove that the locus of **P** is a circle, and find its centre and radius.

17. **A** and **B** are fixed points; **P** moves in such a way that the resultant of forces represented by **PA** and **PB** is always double of the former force. Find the locus of **P**.

18. **OA, OB**, two tangents to a circle, represent two forces in direction and magnitude; if **OC** represent their resultant, and if **C** be on the circumference of the circle, find the inclination of **OA** to **OB**.

19. In a quadrilateral **ABCD**, the sides and diagonals **AB, BC, CD, DA, AC, BD** are bisected at the points **E, F, G, H, K, L** respectively; prove that the resultant of forces represented by **EF, EH, EK, EL** is 2**EG**.

20. **ABCD** is a parallelogram. Prove that the system of forces represented by the lines **PA, BP, PC, DP**, is in equilibrium.

21. **ABC** is a triangle; **G** is the point of intersection of its medians. Prove that the resultant of forces represented by **OA, OB, OC** is represented by 3 **OG** (where **O** is any point).

22. **ABCD** is a quadrilateral; forces are completely represented by the lines **AB, BC, AD, DC**. Prove that their resultant is represented in magnitude and direction by 2 **AC**, and that its line of action bisects **BD**.

23. **ABCD** is a quadrilateral, and **O** is any point in its plane; **E, F, G, H** bisect the sides **AB, BC, CD, DA**, respectively. Prove that the resultant of forces represented by **OA, OB, OC, OD**, is represented by 4 **OK** where **K** bisects **EG**. Deduce that **K** also bisects **FH**.

24. Forces P, Q act along the sides **AB, AC** of a triangle **ABC**. Find the ratio $P : Q$, given that their resultant passes through (i) the orthocentre of the triangle, (ii) the centre of the circumscribing circle.

25. **A, B**, are two fixed points on the circumference of a circle **ABC**. **O** is a point which moves on the arc **ACB**. Forces P, Q, of

constant magnitude, act along **OA, OB.** Prove that the line of action of their resultant intersects the circumference at a fixed point.

26. Forces completely represented by **AB, AC,** and 2 **CB,** act along the sides of a triangle **ABC.** Prove that the line of action of their resultant bisects the side **BC,** and divides the side **AC** in the ratio 2 : 1.

27. **ABCDE** is a regular pentagon, and forces acting at a point are represented in magnitude and direction by the lines **AB, AC, AD, AE, BC, BD, BE, CD, CE, DE** ; prove that the magnitude of their resultant is represented by $4\ \mathbf{AE} + 2\ \mathbf{BD}$.

28. **O** is any point in the plane of a triangle **ABC** ; **D, E, F** are the middle points of the sides. Prove that the resultant of forces represented in magnitude and direction by **OE, OF, DO** is represented by **OA.**

29. In the figure of the last question prove that the forces represented by **OA, OB, OC,** can be replaced by forces represented by **OD, OE, OF.**

30. Forces represented by the sides **AB, AC** of a triangle have a resultant whose line of action passes through the centre of the circumscribing circle. Prove that either **AB** = **AC,** or the angle **BAC** is a right angle.

31. If in Fig. 2 the weight P is 100 grammes, and the angles **HAR, KAR** are each 150°, find graphically or otherwise the weights Q and R.

32. Find the magnitude of the resultants of the following pairs of forces acting at right angles.

Find also the inclination of each resultant to the larger force, either graphically or by means of a table of tangents.

(i) 40 lb., 30 lb.
(ii) 60 lb., 25 lb.
(iii) 5 tons, 3 tons.
(iv) 10 tons, 9 tons.

CHAPTER II.

RECTANGULAR RESOLUTION OF FORCES.

27. **Resolution of Forces.**—In Fig. 1, § 12, we have seen that the two forces **AB, AD** can be replaced by the force **AC.** Conversely the force **AC** can be replaced by the forces **AB, AD**, provided that **AB, AD** are adjacent sides of *any* parallelogram whose diagonal is **AC.** Again in Fig. 9, we have seen (§ 19) that forces, acting at a point, and represented by **AB, BC,** and **CD,** can be replaced by the single force **AD** acting at the same point. Conversely if we take *any* polygon of which **AD** is one side, the single force **AD** could be replaced by a set of forces acting at the same point, and represented by the other sides of the polygon taken in order, starting from **A.**

It is obvious that in the former case an indefinite number of such parallelograms could be drawn, and in the latter case an indefinite number of such polygons.

DEFINITIONS.—If a force is replaced by a set of forces having that force for their resultant, the new forces are called the **components** of the original force; and the process is described as the **resolving,** or the **resolution,** of the original force in its components.

Example.—**AC** and **AD** lie on opposite sides of **AB,** and are inclined to it at angles 90° and 30° respectively (Fig. 21). Resolve a force 10 lb. acting along **AB** into two components acting along **AC** and **AD** respectively.

Draw **AX** of length 10 units to represent the force of 10 lb. Draw **XY** parallel to **DA**, and **XZ** parallel to **CA**. Then since **AYXZ** is a parallelogram, the force **AX** can be replaced by the components **AY**, **AZ**.

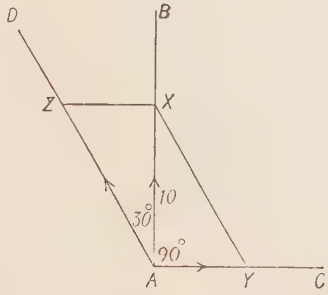


Fig. 21.

Now $\angle \mathbf{AXZ} = 90°$, [Euc. I. 29.]

$$\therefore \frac{\mathbf{AZ}}{\mathbf{AX}} = \sec 30°;$$

$$\therefore \mathbf{AZ} = \mathbf{AX} \cdot \sec 30° = 10 \times \frac{2}{\sqrt{3}} = \frac{20}{\sqrt{3}} = \frac{20}{3}\sqrt{3} = 11{\cdot}55.$$

Also

$$\frac{\mathbf{AY}}{\mathbf{AX}} = \frac{\mathbf{XZ}}{\mathbf{AX}} = \tan 30° : \therefore \mathbf{AY} = \mathbf{AX} \tan 30° = 10 \times \frac{1}{\sqrt{3}} = \frac{10}{3}\sqrt{3}$$

$$= 5{\cdot}77.$$

Thus the required components are 5·77 lb., and 11·55 lb. respectively.

28. Resolutes.—We shall frequently find it convenient to resolve a force into two components along directions which are *at right angles* to each other. Whenever the components are at right angles, they are called **resolutes** or **resolved parts** of the original force.

In the same way the "resolved part" or "resolute" of a force "along a given straight line" means the component of the force along this line *when the other component is at right angles to this line.*

A force P is inclined at an angle A to a given line.

To find the resolutes of P along and perpendicular to that line.

Let **AC** represent the given force P, and let **AX** be the given line; then **CAX** is the angle A.

Draw **AY** perpendicular to **AX**. Draw **CB** and **CD** parallel to **AY** and **AX**; then **AB** and **AD** will represent the required resolutes (§ 27). Let **X**, Y denote these resolutes respectively.

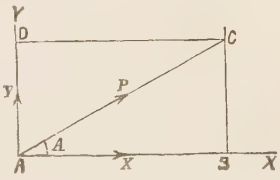


Fig. 22.

Now $\angle \mathbf{ABC} = 90°$;

hence $$\mathbf{AB} = \mathbf{AC} \cdot \frac{\mathbf{AB}}{\mathbf{AC}} = \mathbf{AC} \cos A,$$

and $$\mathbf{AD} = \mathbf{BC} = \mathbf{AC} \frac{\mathbf{BC}}{\mathbf{AC}} = \mathbf{AC} \sin A.$$

$$\therefore X = P \cos A \quad \text{and} \quad Y = P \sin A \ldots\ldots\ldots (1).$$

NOTE.—If (in Fig. 22) P be perpendicular to **AX**, then **P** lies along **AY**, and therefore **B** coincides with **A**; *i.e.* $X = 0$,

Thus, *the resolute of a force along a line perpendicular to itself is zero.*

Conversely, *if the resolute of a force along a given line is zero, the direction of the force is perpendicular to this line.*

Examples.—(1) A man is pulling a trolley along a level road by means of a cord which slopes at 30° to the horizontal. If he pulls with a force of 20 lb., estimate the force urging the trolley forward, and the force tending to lift the trolley.

In Fig. 22 let P represent the pull of the cord on the trolley, viz., 20 lb.; then if **AX** is horizontal, $\mathbf{A} = 30°$. Resolving P into X and Y, X tends only to urge the trolley forward, and Y tends only to lift the front of the trolley from the ground.

Also $$X = P \cos A = 20 \times \frac{\sqrt{3}}{2} = 17{\cdot}32 \text{ lb.}$$

$$Y = P \sin A = 20 \times \tfrac{1}{2} = 10 \text{ lb.}$$

(2) Find the resultant of the following system of forces acting at a point :—2 lb. pulling due N., 3 lb. pulling N.E., 4 lb. pulling S. 30° W.

This system is represented in Fig. 23, where **OX, OY, OX′, OY′** point respectively E., N., W., S.

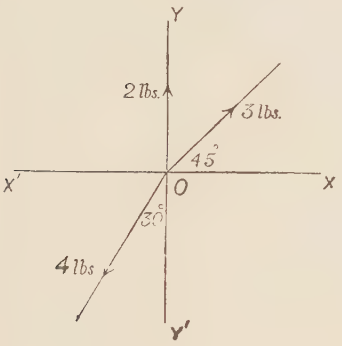


Fig. 23.

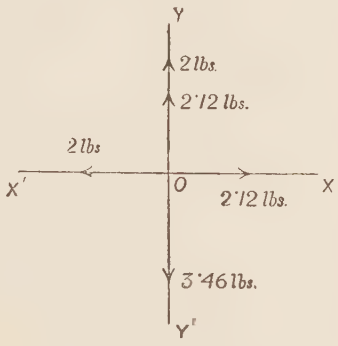


Fig. 24.

Using formula (1) we can replace the force 3 lb. by its resolutes along **OX** and **OY**, and the force 4 lb. by its resolutes along **OY′** and **OX′**.

Thus :—2 lb. N. = 2 lb. N.

3 lb. N.E. = $\begin{cases} 3 \cos 45° \text{ E.} \\ 3 \sin 45° \text{ N.} \end{cases}$ = $\begin{cases} 2{\cdot}12 \text{ lb. E.} \\ 2{\cdot}12 \text{ lb. N.} \end{cases}$

4 lb. S. 30° W. = $\begin{cases} 4 \cos 30° \text{ S.} \\ 4 \sin 30° \text{ W.} \end{cases}$ = $\begin{cases} 3{\cdot}46 \text{ lb. S.} \\ 2 \text{ lb. W.} \end{cases}$

Thus the right hand column gives a system, shown in Fig. 24, which is equivalent to the original system.

Now the east and west forces can be combined by subtraction and give a force ·12 lb. E. ; similarly the north and south forces give a force of 66 lb. N. Thus we have reduced the system to two forces at right angles, which can be reduced to a single force by the method of § 24.

Thus $R = \sqrt{(\cdot 12^2 + \cdot 66^2)} = \cdot 67$,

$\tan \theta = \cdot 66 \div \cdot 12 = 5{\cdot}5$; whence $\theta = 80°$.

Thus the resultant is a force of ·67 lb., acting in the direction 80° north of east, *i.e.* N. 10° E.

29. **The Ship Problem.** We are now in a position to explain how a sailing ship can travel, by "tacking," in the opposite direction to the wind. Fig. 25 gives a "bird's-eye" view of a yacht with a single sail travelling in a north-east direction in an east wind, *i.e.* a wind blowing from the east. (Of course, if the yacht wishes to travel due east it can only do so by "tacking," *i.e.* by travelling alternately in directions which are roughly north-east and south-east respectively.)

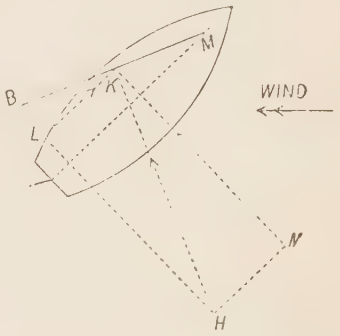


Fig. 25.

BM represents the sail, which is set at such an angle that the rudder side of the sail is exposed to the wind pressure.

The key to the problem lies in the fact that the pressure of the wind on the sail does not act in the direction of the wind, but practically *at right angles to the sail.* This may be roughly explained by saying that the wind can slip along the surface of the sail without friction.

Let **HK** represent the pressure on the sail caused by the wind; then **HK** is $\perp$ to **BM**. Draw **KL** and **HN** parallel to the keel, and **KN** and **HL** perpendicular to the keel. Then by § 27 we may replace **HK** by its resolutes **NK** perpendicular to the keel, and **LK** parallel to the keel.

The resolute **LK** drives the boat rapidly along its keel, in which direction the resistance to motion is small.

The resolute **NK**, though in the diagram (and frequently in practice) it is by far the larger resolute, moves the boat very slowly at right angles to its keel, because the resistance to motion in this direction is very great. It also causes the yacht to "heel over," and necessitates a heavy weight of lead on the keel to prevent the yacht from upsetting.

Thus the motion of the boat is practically along its keel.

30. The algebraic sum of the resolutes of two forces in any direction is equal to the resolute of their resultant in the same direction.

Let **AX** be the given direction, and let forces P, Q and their resultant R be represented as in Figs. 26, 27.

Draw the perpendiculars **BL, CM, DN** on **AX**.

Then **AL, AN, AM** represent the resolutes of P, Q, R.

Now it is easy to prove* that **AN** = **LM**.

Therefore in Fig. 26,

AL+**AN**=**AL**+**LM**=**AM**.

and in Fig. 27,

AL − **AN** = **AL** − **LM** = **AM**.

i.e. in either case, sum of resolutes of P and Q = resolute of R.

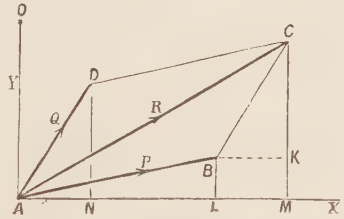


Fig. 26.

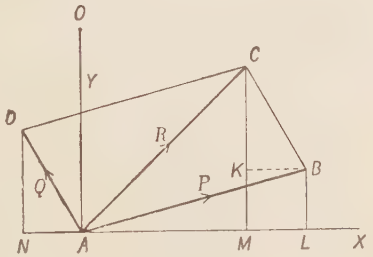


Fig. 27.

Cor. 1. The algebraic sum of the resolutes of *any number of forces* acting on a particle is equal to the resolute of their resultant in the same direction.

This follows at once by applying the theorem successively to the resultant of two of the forces, that compounded of this resultant with a third force, and so on.

Cor. 2. If the system is in equilibrium, the algebraic sum of the resolutes in any direction is zero; for the resultant is zero, and therefore its resolute in any direction is zero.

31. To find the magnitude and direction of the resultant of two forces inclined at a given angle.

Let **AB, AD** represent the two forces P, Q, which are inclined at the angle A (Figs. 28, 29).

* For, drawing **BK** parallel to **AX**, the triangles **DAN, CBK** are equal in every respect; ∴ **AN** = **BK** = **LM**.

Complete the parallelogram **ABCD.** Then **AC** represents the resultant R.

Draw **CM** perpendicular to **AB.**

(i) To find the *magnitude* of the resultant ;

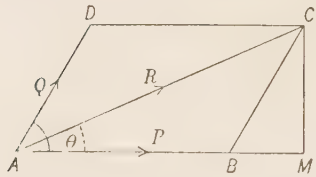


Fig. 28.

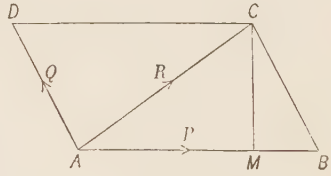


Fig. 29.

Case 1. If A is acute (Fig. 28) :—

$$\mathbf{BC} = \mathbf{AD} = Q\ ; \quad \angle \mathbf{CBM} = \angle \mathbf{DAB} = A\ ;$$

$$\therefore\ \mathbf{BM} = \mathbf{BC} \cdot \frac{\mathbf{BM}}{\mathbf{BC}} = \mathbf{BC} \cdot \cos \mathbf{CBM} = Q \cos A.$$

Thus, applying Euc. II. 12 to the triangle **ABC**,

$$\mathbf{AC}^2 = \mathbf{AB}^2 + \mathbf{BC}^2 + 2\mathbf{AB} \cdot \mathbf{BM}\ ;$$

$$\therefore\quad \boldsymbol{R}^2 = \boldsymbol{P}^2 + \boldsymbol{Q}^2 + 2\boldsymbol{P} \cdot \boldsymbol{Q} \cos \boldsymbol{A} \ldots\ldots\ldots (2)$$

Case 2. If A is obtuse (Fig. 29) :—

$$\mathbf{BC} = \mathbf{AD} = Q\ ; \quad \angle \mathbf{CBM} = 180^\circ - \angle \mathbf{DAB} = 180^\circ - A\ ;$$

$$\therefore\ \mathbf{BM} = \mathbf{BC} \cdot \frac{\mathbf{BM}}{\mathbf{BC}} = \mathbf{BC} \cdot \cos \mathbf{CBM} = Q \cdot \cos (180^\circ - A).$$

Thus, applying Euc. II. 13 to the triangle **ABC**,

$$\mathbf{AC}^2 = \mathbf{AB}^2 + \mathbf{BC}^2 + 2\mathbf{AB} \cdot \mathbf{BM}\ ;$$

$$\therefore\ R^2 = P^2 + Q^2 - 2P \cdot Q \cos (180^\circ - A)$$

$$= P^2 + Q^2 + 2PQ \cdot \cos A,$$

for, by Trigonometry, $\cos (180^\circ - A) = - \cos A$.

Thus we obtain the same formula as in Case 1.

(ii) To find the *direction* of the resultant ;—

Let θ be the inclination of R to P.

Case 1. $$\mathbf{CM} = \mathbf{CB} \cdot \frac{\mathbf{CM}}{\mathbf{CB}} = \mathbf{CB} \cdot \sin \mathbf{CBM} = Q \sin A.$$

$$\mathbf{AM} = \mathbf{AB} + \mathbf{BM} = P + Q \cos A.$$

$$\therefore \tan\theta = \frac{\mathbf{CM}}{\mathbf{AM}} = \frac{Q \sin A}{P + Q \cos A} \quad \ldots\ldots\ldots (3)$$

Case 2. $\mathbf{CM} = \mathbf{CB} \cdot \frac{\mathbf{CM}}{\mathbf{CB}} = \mathbf{CB} \cdot \sin \mathbf{CBM}$

$$= Q \sin (180^\circ - A) = Q \sin A.$$

$$\mathbf{AM} = \mathbf{AB} - \mathbf{BM} = P - Q \cos (180^\circ - A) = P + Q \cos A.$$

$$\therefore \tan\theta = \frac{\mathbf{CM}}{\mathbf{AM}} = \frac{Q \sin A}{P + Q \cos A},$$

the same formula as in Case 1.

Note that if $A = 90^\circ$, we have $\cos A = 0$, $\sin A = 1$; substituting these values in formulae (2) and (3), we obtain formulae (1) and (2) of § 24; as we should naturally expect.

Example.—Find the magnitude and direction of the resultant of forces 5 lb. and 3 lb. inclined to one another at an angle of 60°.

Hence $P = 5$, $Q = 3$, $A = 60^\circ$; thus, by formula (2),

$R^2 = 5^2 + 3^2 + 2 \times 5 \times 3 \times \cos\ 60^\circ = 25 + 9 + 15 = 49$, whence $R = 7$ lb.

Also, by formula (3), $\tan\theta = \frac{3 \sin 60^\circ}{5 + 3 \cos 60^\circ} = \frac{2{\cdot}6}{5 + 1{\cdot}5} = {\cdot}4$; whence $\theta = 22^\circ$. Thus the resultant is a force of 7 lb., inclined to the force 5 lb. at an angle of 22°.

32. When proving formula (1) in § 28 we assumed that A was an acute angle; but the formula is true whatever be the magnitude of the angle A.

Thus suppose that we wish to find the resolved parts of the force P along **OX** and **OY** (fig. 30). Let these resolved parts be X and Y respectively.

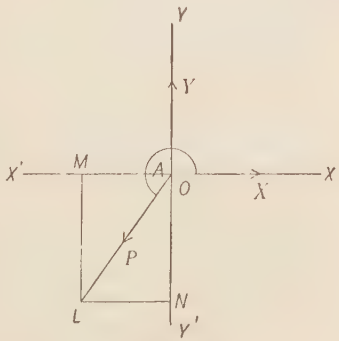


Fig. 30.

Then, in whichever quadrant P lies, the triangle **OLM** may be used as the triangle of reference for determining the trigonometrical ratios of the angle **A**. Then X will always be correctly represented, both in

sign and magnitude, by the base **OM** of this triangle of reference. For X, like **OM**, must be reckoned negative whenever **M** falls to the left of **O**.

But in all cases $\cos A = \mathbf{OM}/\mathbf{OL}$, (provided that the correct sign be assigned to **OM**).

Hence in all cases $\cos A = X/P$; whence $X = P \cos A$; —the sign of X depending on that of $\cos A$. An exactly similar argument applies to Y, which must always be reckoned negative whenever **L** falls below **XX'**.

33. **To find the resultant of any number of forces acting on a particle in one plane.**

Let P_1, P_2, P_3 . . . denote several forces acting at **O**.

Take any direction **OX** in the plane of the forces, and draw **OY** perpendicular to **OX**. Resolve each force into two components in the directions **OX, OY** (§ 28).

Let the components of P_1 be X_1, Y_1, respectively,

" " P_2 " X_2, Y_2, "

and so on; these components being taken with the sign + or −, according to the direction in which they act.

Thus the system of forces is equivalent to forces X_1, X_2, X_3 . . . along **OX**, and Y_1, Y_2, Y_3 . . . along **OY**.

Now these forces may be compounded in any order (§ 19).

Let us first compound together all the forces acting along **OX**. Then, if X denote their resultant, we have

$$X = X_1 + X_2 + X_3 + \ldots$$

Let us next compound together all the forces acting along **OY**. Then, if Y denote their resultant, we have

$$Y = Y_1 + Y_2 + Y_3 + \ldots$$

The whole system of forces is therefore reduced to two known forces—X along **OX**, and Y along **OY**. Hence, if R denote the final resultant of all the forces,

$$R^2 = X^2 + Y^2 = (X_1 + X_2 + \ldots)^2 + (Y_1 + Y_2 + \ldots)^2 \quad \ldots(4).$$

Note that this is the method used in Example (2), § 28.

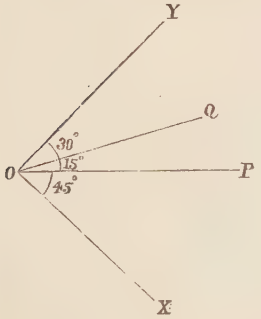
34. *To find the resultant of two forces P, Q, when the angle between their directions is any multiple of* 15°.

The formulae of § 31 enables us to find the resultant for *any* value

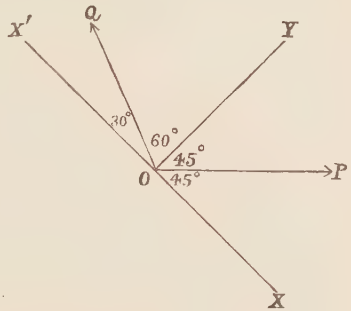
of the angle A, if we find the values of cos A and sin A from the Trigonometrical Tables. If we are not using these tables, we can still readily apply the formulae when A has either of the following values, —0°, 30°, 45°, 60°, 90°, 120°, 135°, 150°, 180°; for the values of cos A and sin A can be then determined by elementary methods.

These include all angles (less than two right angles) that are multiples of 15°, with the exception of 15°, 75°, 105°, 165°. The trigonometrical ratios of these four angles are easily determined by the use of the formulae for the sum or difference of two angles. But the following method enables us to dispense with the aid of even these simple formulae.

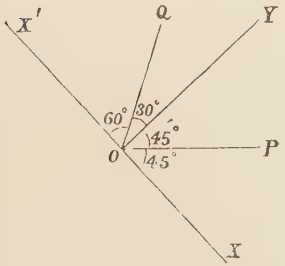
Draw two perpendicular lines **OX**, **OY** inclined at angles of 45° to



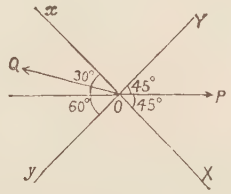
Angle between forces = 15°.
Fig. 31.



Angle between forces = 105°.
Fig. 32.



Angle between forces = 75°.
Fig. 33.



Angle between forces = 165°.
Fig. 34.

the direction of one of the forces, say P. Then it will be seen from Figs. 31-34 that the other force Q makes angles of 30° and 60° with

the lines **OX, OY,** or these lines produced. Hence the forces P, Q can be replaced by their components along **OX, OY,** as in § 28, and their resultant may be found as in § 33.

Example.—To find the resultant of two forces of 4 lb. and 2 lb. inclined at an angle of 165°.

Let **OP, OQ** (Fig. 34) represent the directions of the forces of 2 lb. and 4 lb.

Draw **XOx** making an angle 45° with **OP,** and draw **YOy** perpendicular to **XOx.**

Then $\angle$ **xOQ** = **POQ** − **POY** − **YOx** = 165° − 45° − 90° = 30°,

$\angle$ **QOy** = 60°.

Therefore the components of the force of 4 lb. acting along **OP** are

$$4 \times \sqrt{\tfrac{1}{2}} \text{ or } 2\sqrt{2} \text{ lb. along } \mathbf{OX},$$

$$4 \times \sqrt{\tfrac{1}{2}} \text{ or } 2\sqrt{2} \text{ lb. along } \mathbf{OY};$$

and the components of the force of 2 lb. acting along **OQ** are

$$2 \times \tfrac{1}{2}\sqrt{3} \text{ or } \sqrt{3} \text{ lb. along } \mathbf{Ox},$$

$$2 \times \tfrac{1}{2} \text{ or } 1 \text{ lb. along } \mathbf{Oy}.$$

Therefore the given forces are equivalent to forces X, Y along **OX, OY,** where

$$X = 2\sqrt{2} - \sqrt{3},$$

$$Y = 2\sqrt{2} - 1.$$

Therefore, if R be the required resultant,

$$\begin{aligned} R^2 = X^2 + Y^2 &= (2\sqrt{2} - \sqrt{3})^2 + (2\sqrt{2} - 1)^2 \\ &= 8 - 4\sqrt{6} + 3 + 8 - 4\sqrt{2} + 1 \\ &= 20 - 4(\sqrt{6} + \sqrt{2}) = 4(5 - \sqrt{6} - \sqrt{2}); \end{aligned}$$

$$\therefore \quad R = 2\sqrt{(5 - \sqrt{6} - \sqrt{2})} \text{ lb.}$$

$$= 2{\cdot}13 \text{ lb. approximately (by calculation).}$$

35. **Conditions of equilibrium.**—In order that a system of forces acting on a particle in one plane may be in equilibrium, it is necessary and sufficient that the sums of the resolved parts of the forces along two straight lines at right angles shall be separately zero.

Let **OX, OY** be two straight lines at right angles. Then, by § 33, the resultant R is given by

$$R^2 = X^2 + Y^2,$$

$X = X_1 + X_2 + \ldots =$ sum of resolved parts of forces along **OX**

$Y = Y_1 + Y_2 + \ldots =$,, ,, ,, ,, **OY.**

Now, for *equilibrium*, R must $= 0$. Hence $X^2 + Y^2$ must $= 0$. But square quantities, such as X^2 and Y^2, cannot be negative; hence they must severally be zero. Thus the condition $R = 0$ involves the pair of conditions $X = 0$ and $Y = 0$.

Conversely, if these two conditions are satisfied, $R = 0$ and the forces are in equilibrium.

Observation.—If X were zero and Y were not zero, the forces would have a resultant Y perpendicular to **OX.**

Note.—The resolute of a force in any direction may be said to represent the tendency of the force to produce motion in that direction (cf. § 28, Ex. 1). Thus the algebraic sum of the resolutes of a system of forces in any direction may be said to represent the tendency of the system to produce motion in that direction. Hence the above conditions are equivalent to the statement that the system has no tendency to produce motion along either of the given straight lines.

We may also prove this theorem in a slightly modified form, as follows :—

A system of forces, acting at a point **O**, *will necessarily be in equilibrium, if the algebraic sum of the resolutes of the forces along each of two lines* **OX, OY** *be separately zero, provided that* **OX, OY** *are not in the same straight line.* (Note that **OX, OY** need not be perpendicular.)

If possible, let the system have a resultant R. Then, by § 30, the resolute of R along **OX** is equal to the algebraic sum of the resolutes of the system, and is therefore zero. Thus (by § 28, Note) R must be perpendicular to **OX.** Similarly R must be perpendicular to **OY.** Since it cannot be perpendicular to both lines, there can be no resultant; and therefore the system is in equilibrium.

36. **Lami's Theorem.**—*If three forces keep a particle in equilibrium, each force is proportional to the sine of the angle between the other two.*

Let the forces P, Q, R act along **OP**, **OQ**, **OR**. If they are in equilibrium, their resultant is zero.

$\because$ sum of resolved parts perpendicular to **OR** = resolute of resultant perpendicular to **OR** = 0.

But R has no resolved part perpendicular to **OR**.

Therefore resolved parts of P, Q perpendicular to **OR** are equal and opposite.

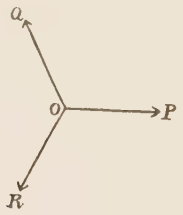


Fig. 35.

$$\therefore\ P \sin \mathbf{ROP} = Q \sin \mathbf{QOR}. \qquad (\S\ 32.)$$

$$\therefore\ \frac{P}{Q} = \frac{\sin \mathbf{QOR}}{\sin \mathbf{ROP}}.$$

Similarly, by resolving perpendicular to **OP**, we have

$$\frac{Q}{R} = \frac{\sin \mathbf{ROP}}{\sin \mathbf{POQ}}.$$

$$\therefore\ \frac{P}{\sin \mathbf{QOR}} = \frac{Q}{\sin \mathbf{ROP}} = \frac{R}{\sin \mathbf{POQ}} \quad \ldots\ldots\ldots (5)$$

as was to be proved.

[This relation is usually deduced from the Triangle of Forces, by means of a well-known theorem in Trigonometry which asserts that in any triangle each side is proportional to the sine of the angle between the other two.]

37. Work.—DEFINITION.—If a particle be moved from **A** to **C** under the action of a constant force P acting parallel to **AB**, and if **CB** be drawn perpendicular to **AB**, then the product

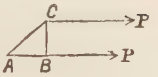


Fig. 36.

$$P \times (\text{distance } \mathbf{AB})$$

is called the **work done** by the force on the particle in changing its position.

The distance moved **AC** is called the *displacement* of the particle, and **AB** is called the *projection* of this displacement on the direction of the force.

If, as in Fig. 37, the direction of the force (**AP**) makes with the direction of the displacement (**AC**) an obtuse angle; then, since the projection **AB** is in the opposite direction to the force P, it is reckoned negative; and accordingly the work done by P is negative.

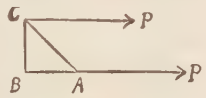


Fig. 37.

OBSERVATION.—This definition holds good whether the displacement **AC** is large or small, provided that the force P remains constant while the particle is moving from **A** to **C**.

The work done by a force is the product of the displacement of its point of application into the resolved part of the force in the direction of that displacement.

In Figs. 36, 37, let $\angle \mathbf{CAP} = \theta$.

Then work done by $P = P \times \mathbf{AB}$

$= P \times \mathbf{AC} \cdot \dfrac{\mathbf{AB}}{\mathbf{AC}} = P \times \mathbf{AC} \cos\theta$

$= P \cos\theta \times \mathbf{AC} = \mathbf{AC} \times$ (resolved part of P along **AC**) [§ 28.

= displacement × resolved part of P along the direction of the displacement.

Note that in applying the preceding argument to Fig. 37, the three quantities **AB**, $\cos\theta$, and the resolute of P along **AC** *must all be reckoned negative, since* θ *is obtuse.*

38. **Principle of work for a single particle.**—*When a particle is moved from one position to another under the action of any number of forces, the algebraic sum of the works done by the several forces is equal to the work done by the resultant.*

For if the particle move from **A** to **C**, then algebraic sum of works done by the several forces

= **AC** × algebraic sum of resolved parts of the several forces along **AC**

= **AC** × resolved part of resultant along **AC** (§ 30)

= work done by the resultant.

39. From this we get the following important corollary :—

Cor. *If a particle, acted on by any number of forces in equilibrium, is moved from one position to another, the algebraic sum of the work done by the several forces is zero.*

For, in this case, the algebraic sum of the resolved parts of the forces along **AC** is zero, whence the result follows at once.

Note.—In §§ 37-39 the student should notice that the forces under consideration are not necessarily the forces which cause the displacement under consideration. For example, in the present section, the system of forces (being in equilibrium) would not cause any motion in the particle, though they would not interfere with any previous motion which the particle may have. Thus the displacement under consideration may be due either to a previous velocity, or to some other force quite distinct from the system under consideration. The same remark must obviously apply to Fig. 37 in § 37, for the force P would tend to prevent the displacement **AC**, not to cause it.

Summary of Results.

Components of P along and perpendicular to line inclined at angle A are $X = P \cos A$, $Y = P \sin A$ . . . . (1), (§ 28.) and X is called the *resolved part* of P along the line. (§ 28.)

Magnitude of the resultant of any two forces P, Q is given by $R^2 = P^2 + Q^2 + 2PQ \cos (P, Q)$ . . . . (2), (§ 31.) where (P, Q) = angle between forces P, Q.

Direction of the resultant is given by

$$\tan \theta = \frac{Q \sin (P, Q)}{P + Q \cos (P, Q)} \quad \text{. . (3).} \quad (\S\ 31.)$$

If R is resultant of forces whose rectangular components are X_1, Y_1; X_2, Y_2, etc.,

$$R^2 = (X_1+X_2+..)^2+(Y_1+Y_2+..)^2 \quad ....(4). \qquad (\S\ 33.)$$

For equilibrium both

$$X_1+X_2+.. = 0 \text{ and } Y_1+Y_2+.. = 0. \qquad (\S\ 35.)$$

If three forces P, Q, R balance,

$$\frac{P}{\sin(Q, R)} = \frac{Q}{\sin(R, P)} = \frac{R}{\sin(P, Q)} \quad ..(5). \qquad (\S\ 36.)$$

Work done by P in displacement **AC**

$= P \times$ (projection of **AC** on P) (§ 37.)

$=$ **AC** $\times$ resolved part of P along **AC**. (§ 37.)

Principle of Work.—Algebraic sum of work of components = work of resultant. (§ 38.)

EXAMPLES II.

(*Selected Problems.*—1, 2, 3, 4, 7, 9, 10, 14, 17, 18, 19, 20, 22, 23, 25, 26, 28.)

1. Resolve the following forces into components, along and perpendicular to the straight lines to which they are inclined at the given angles:

(i.) 4 lb., 30°; (iv.) 3 tons, 90°; (vii.) 8 cwt., 150°;
(ii.) 8 $\sqrt{2}$ oz., 45°; (v.) 12 grammes°, 120°; (viii.) 4 mgr., 180°;
(iii.) 10 kilog., 60°; (vi.) 5 lb., 135°; (ix.) 6 stone, 0°.

2. A force equal to 20 lb. weight, acting vertically upwards is resolved into two forces, one of which is horizontal and equal to 10 lb. weight. What is the magnitude and direction of the other component?

3. A force of $\sqrt{3}$ lb. bisects the angle between two straight lines which include an angle of 60°. Find (i.) the components, (ii.) the resolved parts, of the force along these lines.

4. Find the magnitudes of the resultants of the following pairs of forces inclined at the given angles, namely—

(i.) 3 and 4 lb., 0°; (vi.) 2 and 4 lb., 60°;
(ii.) 10 and 24 grammes, 90°; (vii.) 5 and 10 lb., 120°;
(iii.) 5 and 6 tons, 180°; (viii.) 4 and 12 mgr., 30°;
(iv.) 1 and 3 $\sqrt{2}$ kilog., 45°; (ix.) 4 and 6 oz., 150°.
(v.) 4 $\sqrt{2}$ and 1 cwt., 135°;

5. Two forces of 4 lb. and 10 lb. respectively act at a point and are inclined to each other at an angle of 60°. What is the magnitude of their resultant?

6. Indicate two forces, at right angles to each other, which could maintain equilibrium with the above (see Ex. 5).

7. Show how to find the resultant of two forces represented by the diagonal of a square and one of the sides meeting the diagonal, both acting from the point where they meet. Prove that its magnitude $= \sqrt{5} \times$ a side.

8. A force of 10 lb., acting northwards, is resolved into three components, one of which is 6 lb. north-eastwards, and another 2 lb. westwards. Find the third component.

9. A straight line **COB** has a line **OA** at right angles to it, and forces each of 7 lb. act, one along **OA**, another along **OB**, and a third along the bisector of the angle **COA**. Find the magnitude of the resultant.

10. What is the resultant of three forces, 3, 4, and 5 lb., acting at a point along lines making angles of 120° with each other?

11. If two forces P and Q act upon a particle (i.) when the angle between their directions is 60°, (ii.) when it is 120°, and if R and S are the resultants in these two cases, prove, geometrically if possible, that $$R^2 + S^2 = 2\,(P^2 + Q^2).$$

12. R, S are the smallest and greatest forces which, along with P and Q, can keep a particle at rest. Show that, if P, Q, $\sqrt{RS}$ keep a particle at rest, two of the forces are perpendicular to each other.

13. Two forces, of magnitudes 1 and 3, have a certain resultant when their directions contain a certain angle; the square of the resultant is doubled if the direction of one of the forces is reversed. Find the resolved part of the former force along the direction of the latter.

14. Find the cosine of the angle between the directions of forces of 5 and 7 units, which have a resultant of 8 units. Show that the angle itself exceeds 90°.

15. A force of $10\sqrt{2}$ lb. is inclined at angles 75° and 15° to two perpendicular straight lines. Find the resolved parts of the force along these lines by first replacing it by its components along the internal and external bisectors of the angles between them.

16. Calculate, to two places of decimals, the resultants of $5\sqrt{2}$ and 10 lb. when the angle between them is 15° and when it is 105°; also the resultants of $2\sqrt{2}$ and 3 lb. at an inclination of 75° and also of 165°.

17. If R be the resultant of forces $P + X$ and Y acting at right angles, write down the expression for R^2; and if X, Y are the resolved parts of a force Q along and perpendicular to the direction of P, deduce the formula $R^2 = P^2 + Q^2 + 2PX$.

18. Using a table of tangents, find the angle between the resultant and the larger force in each part of question 4.

19. Two forces of magnitude 3 lb. and 4 lb. have a resultant of magnitude 3 lb. Find, from a table of cosines, the angle between the lines of action of the component forces.

20. Two forces of magnitude 3 lb. and x lb., acting at an angle of 60°, have a resultant of magnitude 5 lb. Find x.

21. Three forces in equilibrium act from the centre of the circumscribing circle of a triangle towards the angular points. Find the ratios of the forces in terms of the sides of the triangle.

22. A force R being resolved into two forces P and Q along lines inclined at angles α and β to the direction of R, find the values of P and Q in terms of α, β, and R.

23. Three forces, each equal to P, act at a point in directions parallel to the sides **AB, BC, CA,** of a triangle. If the sides are proportional to 3, 4, 5, find the resultant.

24. Three forces of magnitude 2, 3, 4, acting at a point, are in equilibrium; find the angles between their lines of action.

25. **AB, BC, CD** are arcs of a circle each subtending an angle of 30° at the centre, and **OA** is a diameter of the circle. If forces be represented by **OA, OB, OC, OD,** and if the resultant of these forces be inclined to **OA** at an angle θ, prove that

$$\tan\theta = (15 + 4\sqrt{3}) / 59.$$

26. If three forces act in given directions at a point **O,** and are in equilibrium; show that, a circle being described through **O** to cut the lines of action in **A, B, C** respectively, the forces are proportional to the sides of the triangle **A, B, C.**

27. Two forces P and Q act at a point along two lines which are inclined at an angle α, and F is their resultant; two other forces P,

and Q' acting along the same lines have a resultant F'. If β is the angle between the lines of action of F and F' prove that

$$\sin \beta = (PQ' - P'Q) \sin \alpha / FF'.$$

28. If the resultant of the two forces is equal to their arithmetic mean, prove that their lines of action are inclined at an angle which is greater than 120°.

29. If, when two equal forces are inclined at an angle 2α their resultant is twice as great as when they are inclined at an angle 2β, prove that

$$\cos \alpha = 2 \cos \beta.$$

30. If two forces are inclined first at an angle of 60°, and afterwards at an angle of 120°, the ratio of the two resultants obtained is $\sqrt{7} : \sqrt{3}$. Find the ratio of the forces.

EXAMINATION PAPER I.

1. What is meant by the *resultant* of two forces, and how can it be determined ?

2. State the proposition known as the " Parallelogram of Forces," and describe an apparatus for verifying it experimentally.

3. Assuming the truth of the " Parallelogram of Forces," enunciate and prove the proposition known as the " Triangle of Forces."

4. Forces of 2, 3, and 4 lb. act at a point **O** in directions parallel to the sides **AB, AC, BC** of an equilateral triangle, respectively. Find their resultant.

5. State the proposition known as the " Polygon of Forces." How far is the converse true ?

6. If **P** be a point in a straight line **AB** such that $m.\mathbf{AP} = n\ \mathbf{PB}$, and if **O** be any other point, prove that two forces represented by $m.\mathbf{OA}$ and $n.\mathbf{OB}$ have a resultant represented by $(m + n)\ .\ \mathbf{OP}$.

7. Show that a force may be resolved into two components in any number of different ways, and explain what is meant by the *resolved part* of a force in any given direction.

8. **ABCD** is a parallelogram ; **E** is a point in **CD**. Find, when possible, a point **F** in **AB** such that the magnitude of the resultant of two forces represented by **AE**, **AF**, may be represented in magnitude by **AD**.

9. State and prove the formula giving the magnitude of the resultant of any two forces in terms of the components and the resolved part of one component along the line of action of the other.

10. Three forces act at a point and keep equilibrium ; find their ratios, having given the angles between them.

CHAPTER III.

EASY PROBLEMS ON EQUILIBRIUM.

40. **On the nature of forces.**—A little reflection will convince the student that our conception of **force** is very indefinite. We probably derive our initial idea of force from the feeling of *effort* which we experience when we move, or interfere with the motion of, heavy bodies. But when we come to consider by what means one body can "exert a force" on another, we find our conceptions very unsatisfactory. For instance, if we pull a trolley by means of a rope, we understand that the trolley follows us because the rope refuses either to stretch considerably or to break; further, we say that the rope does not stretch because it is not sufficiently "elastic," and does not break because of the "adhesiveness" of the particles of which it is composed, *i.e.* their tendency to adhere to one another. But we are frankly ignorant as to the reasons why the rope is inelastic, or why the particles of the rope cohere.

Again, the earth attracts the moon, or a magnet attracts a piece of steel, although there is no apparent connection between them. It is supposed that there *is* a connection between them; that all space is filled with some kind of substance (not matter) which we call "ether"; which substance is answerable for all action at a distance; it transmits light and heat, and produces gravitational attraction, and electric and magnetic phenomena. It is practically certain that in broad outline this theory is correct; but the details are at present very uncertain, and we cannot say to what extent they will prove to be beyond our powers of comprehension or of discovery.

Types of forces. We can distinguish various types of Forces, such as **thrusts, tensions, attractions, repulsions.**

A **thrust** is an action which takes place between bodies *in contact* tending to prevent them from penetrating into one another; for instance, there is a thrust between a ladder and the wall against which it is leaning; and between the ladder and the ground on which it is standing.

Such an action is often called a "pressure"; but in the strictly scientific use, the word pressure is restricted to the measurement of "thrust per unit area," such as the "atmospheric pressure" of 14·7 *lb. per square inch* or the water pressure on the bed of a river of (perhaps) 1000 *lb. per square foot*, etc.

A **tension** is an action which takes place between bodies which are fastened together, tending to prevent them from separating. For instance, the action between the rope and the trolley in § 40, or between two parts of the rope, is called a tension.

Attractions and **repulsions** are instances of actions between two bodies which are not in contact. In the case of attraction the bodies tend to approach one another; in the case of repulsion they tend to separate from one another. Every particle of matter attracts every other particle, with more or less force according to the distance between them; bodies which are similarly electrified repel one another, and bodies which are dissimilarly electrified attract one another, and so on.

Strain.—It is worthy of note that force is accompanied by *strain*, *i.e.* by change of shape in the bodies concerned. Thus when the ladder leans against the wall, both the ladder and the wall are slightly compressed in the region of contact; when the rope pulls the trolley both the rope and the trolley are slightly elongated; and in all cases of attraction and repulsion it is probable that the action is produced by some state of strain in the intermediate ether. Whether, in any particular case, it is the force which produces the strain, or the strain which produces the force, is a doubtful question; perhaps it is also a meaningless question. The essential fact is that force and strain are inseparable.

Action and reaction.—It is particularly important to notice that *the action between two bodies is always mutual.* Newton states this in his Third Law, " Action and reaction are equal and opposite ; " that is to say, if a body **A** exerts a force on a body **B,** then the body **B** always exerts an equal and opposite force on the body **A.** For example, if a book is lying on a table, there is an action between the book and the table *which has a definite effect on each.* If we are considering the effect on the table, we say that the book presses down on the table ; if we are considering the effect on the book, we say that the table supports the book. These are what Newton calls the action and reaction ; we may conceive them, for our own purpose, as two separate and distinct forces ; they are in reality inseparable parts of the one action, due to the contact and its circumstances.

Again, the attraction between a magnet and a piece of steel is easily proved to be mutual. If a magnet and a piece of steel, of equal mass, were placed on two little trolleys, so that each was free to approach the other, we should find that they moved equally toward each other, and met half-way. The attraction between the earth and the moon is mutual ; the effect on the moon is much greater than the effect on the earth, because the mass of the moon is much less than that of the earth ; but the mutual action, whatever be its cause, produces an equal force on each.

41. **On the statical hypotheses.**—For purposes of mathematical calculation it is obviously necessary to begin with simple conceptions. We shall therefore, in the present text-book, not only omit all considerations as to the ultimate nature of force, but also omit all considerations as to the relation between force and strain. We shall suppose that bodies are **rigid,** *i.e.* that there is no change of shape or size, whatever forces may be applied to the body (a supposition which is approximately correct with regard to many solid bodies under ordinary circumstances). We shall also consider for the rest of this chapter that the bodies are sufficiently small to be treated as particles ; *i.e.* that the forces acting on the body may be regarded as acting at a point.

The fundamental ideas on which elementary Statics is based are (i) *the Parallelogram of Forces and* (ii) *Newton's Third Law.*

41*a*. Let us consider a few simple cases of the statical method of investigating force problems :—

(1) Let us consider a book lying on a table and an ink-pot standing on the book (Fig. 38).

There is an action between the ink-pot and the book, which in accordance with Newton's Third Law we regard as two forces equal in magnitude and opposite in direction, viz. R' the downward thrust of the ink-pot on the book, and R the "reaction" of the book,—an equal upward thrust supporting the ink-pot.

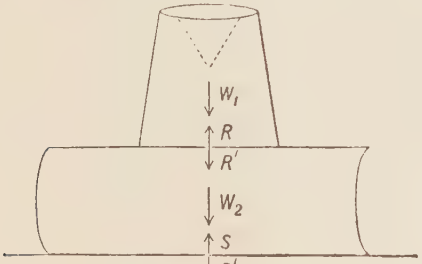


Fig. 38.

There is an action between the book and the table, which we regard as S'—the downward thrust of the book on the table, and S the "reaction" of the table,—an equal upward thrust supporting the book.

Also both ink-pot and book are acted on by their weights, W_1 and W_2, downward forces due to the earth's attraction. The "reactions" of these forces are the forces with which the ink-pot and book attract the earth upwards, which have no effect on the earth owing to its immense mass.

The two forces acting on the ink-pot are in equilibrium (since the ink-pot is at rest), hence $R = W_1$. But by Law III., $R = R'$; hence $R' = W_1$. Thus the downward thrust of the ink-pot on the book is equal to the weight of

the ink-pot (a fact which we generally assume as self-evident).

The three forces acting on the book are in equilibrium, viz. R', W_2, and S. Since the first two forces act vertically down, the third must act vertically up and must be equal to their sum; *i.e.* $S = W_2 + R'$. But by Law III. $S' = S$; hence $S' = W_2 + R' = W_2 + W_1$. Thus the downward thrust of the book on the table is equal to the sum of the weights of the book and the ink-pot. And so on.

(2) Let us consider the case of a bottle hanging from a nail by a piece of string, the nail fastened into a wall (Fig. 39).

We usually neglect the weight of the string and regard it merely as a connection between the bottle and the nail; that is to say, there is an action between the bottle and the nail, transmitted through the medium of the string; the nail exerts an upward pull T on the bottle, which supports it; the bottle exerts an equal downward pull T on the nail. Each of these forces is usually referred to as the "tension" of the string, as it is transmitted by means of the string.

Fig. 39.

Since the bottle is in equilibrium, the "tension of the string" at the bottle is an upward force equal to the weight of the bottle. Hence, by the preceding explanation, the "tension of the string" at the nail is a downward force equal to the weight of the bottle. Since the nail is at rest, the tension of the string at the nail, the weight of the nail, and the complicated set of forces exerted on the nail by the wall, together form a system in equilibrium.

In the same way, whenever a stretched light string connects two bodies, their mutual action is, in accordance with Newton's Third Law, represented by two equal forces, one on each body, acting along the line of the string toward the other body; each of which forces is referred to as the "tension" of the string.

We have considered these simple cases at considerable length because in all statical problems it is very important

that the student should clearly grasp the application of Newton's Third Law;—that the action between any two bodies is to be conceived as consisting of two equal and opposite forces, one acting on each body; and that the equilibrium of any one body is produced by the forces which the other bodies exert on this body, and *not by the forces which this body exerts on the other bodies.* Thus in dealing with the equilibrium of the book in Example 1, we are concerned with the downward thrust of the ink-pot on the book, but not with the upward thrust of the book on the ink-pot.

42. **On smooth and rough surfaces.**

DEFINITIONS.—A **smooth surface** is one which can only exert a reaction at right angles to itself.

A **rough surface** is one which can exert a reaction which is not at right angles to itself.

That these definitions are reasonable will easily be seen from the following considerations :—

Suppose a body to be placed on a board, and suppose

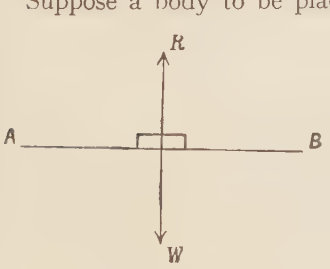


Fig. 40.

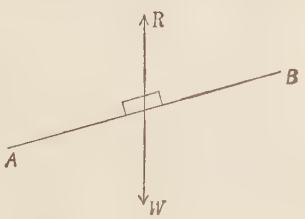


Fig. 41.

the board to be gradually tilted up at one end (Figs. 40 and 41). When the board is tilted far enough the body will begin to slide down; and the rougher the board, the farther must the board be tilted before sliding takes place. A perfectly smooth board could not be tilted at all, without the body sliding down it. Now the only forces acting on the body are its weight W acting vertically down, and the "reaction" of the plane R, which tries to support the body.

Thus the body will not be in equilibrium unless R is equal to W and acts vertically up. Thus the rough board in Fig. 41 must be able to exert a "reaction" R which is not perpendicular to the board; the smooth board in Fig. 40 can exert a reaction R perpendicular to itself, but could not support the body when tilted as in Fig. 41, since this requires a reaction which is not perpendicular to the board.

43. **Solution of problems by the method of force triangles.**—When a body is in equilibrium *under three forces* we can find the ratios of these forces by means of the *converse of the "Triangle of Forces"* (§ 17). Hence if one of the forces is known, the others can be determined. The following examples will illustrate the method.

44. **A body is in equilibrium on a smooth inclined plane under the action of a horizontal force. Find the magnitude of this force and the pressure on the plane.**

Let **AC** (Fig. 42) represent the smooth inclined plane; **O** the body which is at rest on the plane, and prevented from slipping down the plane by a horizontal force P. Draw **AB** horizontal, and **BC** vertical. Let the inclination of the plane to the horizontal (*i.e.* the angle **CAB**) be i.

Now there are three forces acting on the body, viz. P, W its weight acting vertically down, and R the reaction of the plane. Since the plane is smooth we assume that R is perpendicular to **AC** (§ 42). Then these three forces are respectively perpendicular to the sides of the triangle **ABC**; viz.:

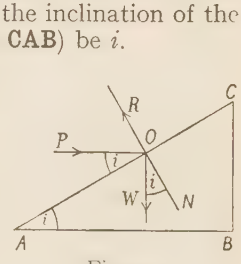


Fig. 42.

$$\left.\begin{array}{ll} P \text{ is perpendicular to } & \mathbf{BC}, \\ R \quad\quad ,, \quad\quad ,, & \mathbf{CA}, \\ W \quad\quad ,, \quad\quad ,, & \mathbf{AB}. \end{array}\right\}$$

Hence by § 22, section (5), $P : R : W = \mathbf{BC} : \mathbf{CA} : \mathbf{AB}$.

Since $P : W = \mathbf{BC} : \mathbf{AB}$; $\therefore P = W \cdot \dfrac{\mathbf{BC}}{\mathbf{AB}} = W \tan i$ ..(1)

Since $R : W = \mathbf{CA} : \mathbf{AB}$; $\therefore R = W \cdot \dfrac{\mathbf{CA}}{\mathbf{AB}} = W \sec i$ (2)

The lines **AC, AB, BC** are often called the "length, base, and height" of the plane respectively, and are represented in formulae by the letters l, b, and h respectively. Thus the ratios of the three forces can be expressed by the formula

$$P : R : W = h : l : b \quad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

whence $$P = W \cdot \frac{h}{b}; \; R = W \cdot \frac{l}{b}.$$

The *pressure on the plane* is equal and opposite to R.

45. **A body is in equilibrium on a smooth inclined plane under the action of a force acting up the plane. Find the magnitude of this force and the pressure on the plane.**

Let **AC** (fig. 43) represent the smooth inclined plane, **O** the body which is at rest upon it, supported by the force P acting upon the plane. Draw **AB** horizontal and **CB** vertical. Draw **BD** perpendicular to **AC**.

Then $\angle$ **DBC** $= \angle$ **BAC**; for each is complementary to the $\angle$ **ABC**.

$\therefore \quad \angle$ **DBC** $= i$.

Two other forces are acting on the body, viz. W its weight acting vertically downwards, and R the reaction of the plane perpendicular to **AC** (§ 42). Now the three forces P, R, W, are respectively parallel to the three sides of the triangle **BCD**, viz.

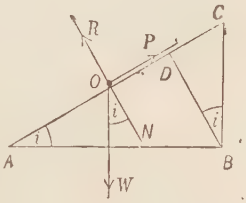


Fig. 43.

$$\left.\begin{array}{l} R \text{ is parallel to } \mathbf{BD} \\ P \;\; ,, \quad ,, \quad ,, \; \mathbf{DC} \\ W \;\; ,, \quad ,, \quad ,, \; \mathbf{CB} \end{array}\right\}$$

Thus $$R : P : W = \mathbf{BD} : \mathbf{DC} : \mathbf{CB} \qquad [\S\ 17.$$

Since $P : W = \mathbf{DC} : \mathbf{CB}$, $\therefore \; P = W \cdot \frac{\mathbf{DC}}{\mathbf{CB}} = W \sin i \quad \ldots (4)$

Since $R : W = \mathbf{BD} : \mathbf{CB}$, $\therefore \; R = W \cdot \frac{\mathbf{BD}}{\mathbf{CB}} = W \cos i \quad \ldots (5)$

Again, the triangle **CDB** is equiangular to the triangle

CBA ; hence the sides of **CDB** are proportional to the sides of **CBA** ;

i.e. $\mathbf{BD} : \mathbf{DC} : \mathbf{CB} = \mathbf{AB} : \mathbf{BC} : \mathbf{CA} = b : h : l.$

Thus the ratios of the forces can be expressed by the formula, $R : P : W = b : h : l$ ..... (6)

The *pressure on the plane* is equal and opposite to R by Law III.

Examples.—(1) In the crane **ACB** (Fig. 44) the jib or rod **CA** is 12 ft. long, and is connected to the wall **BC** by a chain **AB**, 8 ft. long, attached to a point **B** 6 ft. above **C**. Find the thrust in the rod and the tension in the chain.

We will neglect the weight of **CA** (which is probably small compared to the other forces) ; we can then regard the rod **CA** as a mere connection between **C** and **A**. Thus **C** is acting on **A** with a thrust P transmitted along **CA**.

Now *consider the equilibrium of the point* **A**. It is acted on by three forces, viz. P the thrust of the rod **CA**, T the tension of the chain **AB**, and W (= 18 cwt.) the pull of the weight vertically downwards (transmitted by a chain). These three forces must be in equilibrium.

But these three forces are respectively parallel to the sides of the triangle **ABC**, for

P acts along **CA***
T acts along **AB***
W acts parallel to **BC**

$\therefore\ P : T : W = \mathbf{CA} : \mathbf{AB} : \mathbf{BC}$ [§ 17.

Thus $P : W = \mathbf{CA} : \mathbf{BC}$; or

P : 18 cwt. = 12 ft. : 6 ft. ;

whence $P = 18 \times 12 \div 6$ or 36 cwt.

Also $T : W = \mathbf{AB} : \mathbf{BC}$; or

T : 18 cwt. = 8 ft. : 6 ft.

whence $T = 18 \times 8 \div 6$ or 24 cwt.

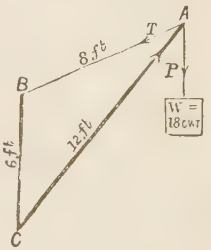


Fig. 44.

(2) **AB** and **AC** are two chains 9 ft. and 12 ft. long attached to pegs **B**, **C** at a horizontal distance of 15 ft. apart. To find the pulls in the chains when a weight of 1 ton is suspended from **A**.

Here the lengths **AB**, **AC**, **BC** are proportional to 3, 4, 5 ;

$\therefore\ \mathbf{BC}^2 = \mathbf{BA}^2 + \mathbf{AC}^2$;

therefore **BAC** is a right angle (by Euc. I. 48).

* The validity of §§ 16, 17 is not affected if P acts along **BC** (Figs. 4, 5) instead of parallel to **BC**.

The point **A** is in equilibrium under three forces :—P the pull of the chain **AB**, Q the pull of the chain **AC**, and W(= 1 ton) the pull of the weight.

We can use **ABC** as a perpendicular triangle of forces.

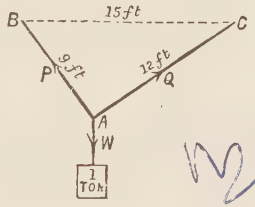


Fig. 45.

For P acts at right angles to **CA**
Q ,, ,, ,, ,, **AB**
W ,, ,, ,, ,, **BC**

Hence $P : Q : W = \mathbf{CA} : \mathbf{AB} : \mathbf{BC}$
$= 12 : 9 : 15 = 4 : 3 : 5.$

Thus
$P = \frac{4}{5}W = \frac{4}{5}$ ton ; $Q = \frac{3}{5}W = \frac{3}{5}$ ton.

(3) A weight of 1 ton is attached at **B** to a rod **AB**, which is drawn aside from the vertical position through 30° by a chain **BD** attached to B. Find the pull in the rod, supposing **BD** to make an angle of 60°, (i.) with the upward drawn vertical, (ii.) with the downward drawn vertical.

The point **B** is in equilibrium under the following three forces—R the required pull in the rod **BA**, P the pull in the chain **BD**, Q (= 1 ton) the given weight.

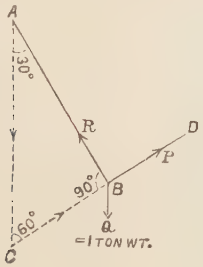


Fig. 46.

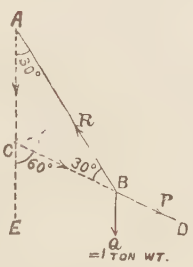


Fig. 47.

Take any point **A** on the rod, and let the vertical through **A** meet **DB** produced in **C**.

P acts along **CB**
R acts along **BA**
Q acts parallel to **AC**

$\therefore P : R : Q = \mathbf{CB} : \mathbf{BA} : \mathbf{AC}.$

(i.). In Fig. 46, $\angle BAC = 30°$, $\angle BCA = 60°$; hence $\angle ABC = 90°$

Hence, by Trigonometry, $CB : BA : AC = 1 : \sqrt{3} : 2$.

$$\therefore P : R : Q = 1 : \sqrt{3} : 2. \quad \therefore R = Q\frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2} \text{ ton.}$$

(ii.) In Fig. 47, $\angle BAC = 30°$, $\angle BCE = 60°$,

$\therefore \angle ABC = 30°$. [Euc. I. 32.

Therefore **ABC** is an isosceles triangle having its angles equal to 30°, 30°, 120° respectively; hence if **C** be joined to the middle point of **AB**, the triangle will be divided into two triangles in each of which the angles are 30°, 60°, 90° respectively.

Thus, by Trigonometry, $AC : CB : BA = 2 : 2 : 2\sqrt{3} = 1 : 1 : \sqrt{3}$.

Thus $P : R : Q = CB : BA : AC = 1 : \sqrt{3} : 1$;

whence $R = Q \cdot \sqrt{3} = \sqrt{3}$ tons.

Note that in all the preceding examples we have found or constructed a "force triangle," *i.e.* a triangle whose sides are either respectively parallel or respectively perpendicular to the lines of action of the three forces.

46. **Equilibrium on an inclined plane, the supporting force being applied in any direction whatever.**

When the supporting force P is applied in any direction other than those considered above, its magnitude can, in general, only be calculated by Trigonometry, but it may be determined graphically thus :—

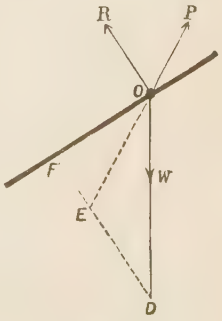


Fig. 48.

On the vertical through **O** measure **OD** downwards containing as many units of length as there are units of force in the weight W. Draw **DF** perpendicular to the inclined plane, and let it meet the line in which P is applied in the point **E**. Then, by the Triangle of Forces, **EO** represents the force P, and **DE** represents the reaction R. Hence, if the figure is carefully drawn, P and R can be found by measuring the lengths **EO, DE.**

For different directions of P, the point **E** always lies on the straight line **DF.** Evidently **EO** is least when **E** is at **F,** because the perpendicular **OF** is less than any other straight line drawn from **O** to the line **DF.**

47. **Solution of problems by the method of resolution of forces.**—We obtain another important method for solving problems by using the theorem of § 35, viz. that if a system of forces acting at a point is in equilibrium, the algebraic sum of the resolutes of the forces in any direction is equal to zero.

As examples of this method we will work again the two cases of equilibrium of a body on a smooth inclined plane, which we have already worked in §§ 44, 45.

(1) If the force P is horizontal.

Since the arms of the angle **NOW** are respectively perpendicular to those of the angle **CAB**,

$$\therefore \angle \mathbf{NOW} = \angle \mathbf{CAB} = i.$$

Also, by parallels,

$$\angle \mathbf{POA} = \angle \mathbf{OAB} = i.$$

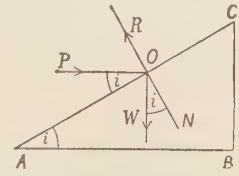


Fig. 48*a*.

Now the forces P, R, W form a system in equilibrium; hence by § 35 the algebraic sum of the resolutes of these forces along **AC** is zero.

Now the force R is perpendicular to **AC**, hence its resolute along **AC** is zero (§ 28, Note). The force P is inclined to **AC** at an angle i; hence its resolutes along **AC** is $P \cos i$ (§ 28). The force W is inclined to **CA** at an angle $(90° - i)$, hence its resolute along **CA** is $W \cos (90 - i)$, by § 28, which by Trigonometry reduces to $W \sin i$.

Also since the resolute of P acts *up* the plane and the resolute of W acts *down* the plane, the "algebraic sum" of the resolutes will be their difference. Hence we obtain

$$P \cos i - W \sin i = 0$$

whence $$P = W \tan i$$

Again, suppose **WO** is produced to **M**; then the algebraic sum of the resolutes of the forces along **WM** is zero. *W acts along this line*, and its resolute is W *downwards*. P is perpendicular to this line; hence its resolute is zero. R is inclined to **OM** at an angle i; hence its resolute is $R \cos i$ *upwards*. Thus we obtain the equation $R \cos i - W = 0$, whence

$$R = W \sec i.$$

These results are in agreement with § 44.

(2) If the force P acts up the plane ;—

Again, P, R, W form a system in equilibrium.

Resolve along **AC** ; the resolute of R is zero, of P is P, and of W is $W \cos (90 - i) = W \sin i$.

Thus, by § 35, taking account of the opposite direction of the resolutes, $P - W \sin i = 0$;
i.e. $P = W \sin i$.

Resolve along **NR** ; the resolute of R is R, of P is zero, and of W is $W \cos i$.

Thus, by § 35, taking account of the directions of the resolutes, $R - W \cos i = 0$; *i.e.* $R = W \cos i$.

Thus we obtain the same result as in § 45.

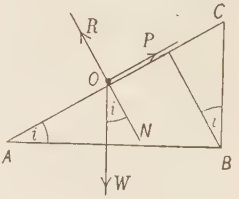


Fig. 48*b*.

Example.—A string is attached to two pegs **A**, **C** in a horizontal line 24 feet apart, and a weight of 10 lb. is suspended from its middle point **B**. If this point falls 5 feet below the line **AC**, to find the tension in the string.

Draw **BE** perpendicular to **AC**, and therefore vertical.

We can easily show that :—

$\angle$ **BAE** = $\angle$ **BCE** (Euc. I. 5) ;

$\triangle$ **BAE** $\equiv$ $\triangle$ **BCE** ;

AE = $\frac{1}{2}$**AC** = 12 ft. ;

AB = **BC** = 13 ft. (Euc. I. 47).

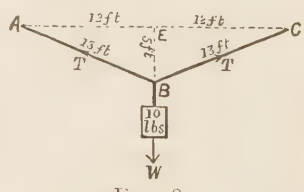


Fig. 48*c*.

Now the knot **B** is in equilibrium under three forces, viz. the two tensions along **BA** and **BC** and the downward pull of the weight. *We may assume that the two tensions are equal* ; for the line **EW** divides the figure into two halves which are exactly similar ; hence any calculation which we can make with regard to any magnitude on the left side of the figure will also apply to the corresponding magnitude on the right side of the figure.*

Thus the forces in equilibrium at **B** are T along **BA**, T along **BC**, and 10 lb. vertically down.

---

* This argument from " symmetry " is constantly occurring. The line **EW**, which divides the figure into two exactly similar halves, is called the " axis of symmetry." The figure is said to be " symmetrical about the line **EW**." Note that any line which joins two *corresponding* points is bisected at right angles by the axis of symmetry.

Resolving vertically, *i.e.* along the line **WE**, we have by § 35

$$T \cos \mathbf{ABE} + T \cos \mathbf{CBE} - 10 = 0,$$

i.e. $T \cdot \frac{5}{13} + T \cdot \frac{5}{13} - 10 = 0, \quad \left(\text{for } \cos \mathbf{ABE} = \frac{\mathbf{BE}}{\mathbf{BA}} = \frac{5}{13}\right),$

whence $T = 13$ lb.

47*a*. It is obvious that the method of solving problems by "force triangles" can only be applied to a system of *three* forces in equilibrium at a point. The method of "resolution of forces" has the advantage that it can be applied to *any number* of forces in equilibrium at a point.

Example.—A mass of 2 lb. is supported on a smooth plane, inclined at 60° to the horizontal by two equal forces; one acts upon the plane, the other acts horizontally. Find their magnitude. (See Fig. 48*d*.)

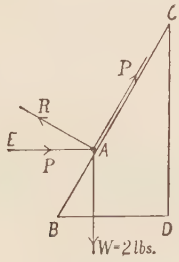


Fig. 48*d*.

The mass at **A** is in equilibrium under the following set of forces: its weight W vertically down, the reaction of the plane R at right angles to **BC**, and the two equal forces P horizontal and P along **AC**.

Resolve along **BC**. R is perpendicular to **BC**, hence its resolute is zero; the horizontal force P is inclined at 60° to **BC** (**EAB** = 60°, I. 29), hence its resolute is $P \cos 60°$; the other force P acts along **BC**, hence its resolute is P; the vertical force W is inclined at 30° to **CB**, hence its resolute along **BC** is $-2 \cos 30°$. Thus we have by § 35

$$P \cos 60° + P - 2 \cos 30° = 0,$$

i.e. $P \cdot \frac{1}{2} + P - 2 \cdot \frac{\sqrt{3}}{2} = 0$, whence $\mathbf{P} = \frac{2}{3}\sqrt{3}$ lb.

By resolving in some other direction we could obtain an equation to find R.

48. **On the method of solution by Lami's Theorem.** If we have only three forces acting at a point, it may be more convenient to use Lami's Theorem (§ 36) than the triangle of forces.

As an illustration of this method let us solve again Example 3, § 45.

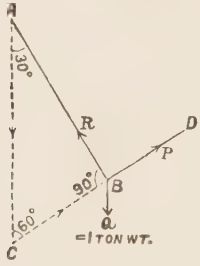


Fig. 48*e*.

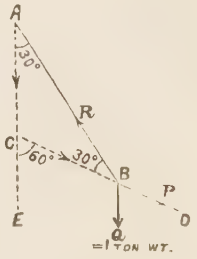


Fig. 48*f*.

In Fig. 48*e* since the forces P, Q, R at **B** are in equilibrium, we have

$$\frac{P}{\sin \mathbf{QBR}} = \frac{Q}{\sin \mathbf{RPB}} = \frac{R}{\sin \mathbf{PBQ}}.$$

Now $\angle$ **QBC** = 60° (by parallels); hence $\angle$ **QBP** = 120° and $\angle$ **QBR** = 150°. Also $\angle$ **PBR** = 90°. Thus we have

$$\frac{P}{\sin 150°} = \frac{Q}{\sin 90°} = \frac{R}{\sin 120°},$$

or

$$\frac{P}{\frac{1}{2}} = \frac{Q}{1} = \frac{R}{\frac{\sqrt{3}}{2}}.$$

Thus

$$P = \tfrac{1}{2}Q = \tfrac{1}{2} \text{ ton wt.}$$

$$R = \frac{\sqrt{3}}{2}Q = \frac{\sqrt{3}}{2} \text{ ton wt.}$$

Again, in Fig. 48*f*, $\angle$ **PBR** = 150°;

$\angle$ **PBQ** = 60° (by parallels); thus $\angle$ **QBR** = 150°. Hence by Lami's Theorem,

$$\frac{P}{\sin 150°} = \frac{Q}{\sin 150°} = \frac{R}{\sin 60°},$$

i.e.

$$\frac{P}{\frac{1}{2}} = \frac{Q}{\frac{1}{2}} = \frac{R}{\frac{\sqrt{3}}{2}}.$$

Thus $P = Q = 1$ ton wt.; $R = Q\sqrt{3} = \sqrt{3}$ tons wt.

Example.—**ABC** is an equilateral triangle formed of weightless inextensible strings; the side **AB** is horizontal; **A** and **B** are tied to fixed points **D** and **E**, by equal weightless inextensible strings **AD**, **BE**; a weight of 2 lb. is attached at **C**. Given that the angles **DAB**, **ABE** are each 150°, find the tension in every string.

At each of the knots **A, B, C** *we have a set of three forces in equilibrium*, viz. the tensions of the three strings which meet at the knot. *In any string the tensions at the two extremities are equal in magnitude and opposite in direction* [§ 41*a*]: such as the pairs (S, S), (Q, Q), (T, T) in the figure.

Draw **CH** vertical; then, since **AB** is horizontal, we have

$\angle$ **CHA** $= 90°$; $\angle$ **HCA** $= 30°$;

$\angle$ **ACF** $= 150°$; $\angle$ **HCB** $= 30°$;

$\angle$ **FCB** $= 150°$;

$\angle$ **DAC** $= 360° - 150° - 60°$

$= 150°$;

$\angle$ **EBC** $= 150°$.

Applying Lami's Theorem to the forces acting at **C**,

$$\frac{S}{\sin \mathbf{FCB}} = \frac{2}{\sin \mathbf{ACB}};$$

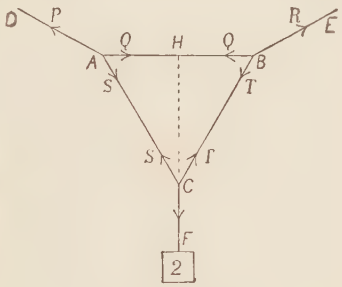


Fig. 48*g*.

i.e.

$$S = \frac{2 \sin \mathbf{FCB}}{\sin \mathbf{ACB}} = \frac{2 \sin 150°}{\sin 60°}$$

$$= 2 \times \frac{1}{2} \times \frac{2}{\sqrt{3}} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3} \text{ lb.}$$

Applying Lami's Theorem to the forces acting at **A**,

$$\frac{P}{\sin 60°} = \frac{Q}{\sin 150°} = \frac{S}{\sin 150°};$$

but $S = \frac{2\sqrt{3}}{3}$; whence $P = \frac{S \cdot \sin 60°}{\sin 150°} = \frac{2\sqrt{3}}{3} \times \frac{\sqrt{3}}{2} \times \frac{2}{1} = 2$ lb

and

$$Q = S = \frac{2\sqrt{3}}{3} \text{ lb.}$$

Also the figure is symmetrical about the line **FH** (see note on p. 56),

hence

$$T = S \frac{2\sqrt{3}}{3} \text{ lb.}$$

$$R = P = 2 \text{ lb.}$$

Note particularly the method of the preceding problem ;– From the equilibrium of the forces at **C** we can determine the magnitude of S at **C** ; this gives us the magnitude of S at **A** (Law III) ; from the equilibrium of the forces at **A** we can determine the magnitude of P and Q.

48*a*. In solving any mathematical problem it is important to remember that the **number of independent equations must in general be equal to the number of unknowns.** Thus to find three unknowns x, y, z, we require three independent simultaneous equations, *i.e.* three equations no one of which can be derived from the others.

In Mechanics **the conditions of equilibrium of a set of forces at a point are equivalent to two independent equations,** for we may either resolve in two directions (applying § 35), or use the triangle of forces (§ 17), or Lami's theorem (§ 36). Either method gives two independent equations.

Thus in the example in § 47*a* there are only two unknowns, P and R. We obtain two independent equations by resolving in different directions and thus determine both the unknowns.

If we had not known that the horizontal force and the force up the plane were equal, we should have had three unknowns P, P', and R. The problem could not then have been solved without further data, as we can only obtain two independent equations by resolving. We could of course resolve in three different directions, and so obtain three equations ; but they would *not be independent* equations, and the attempt to solve them for the three unknowns would fail. *Any values of* P, P', R which satisfied the first two equations would necessarily make with W a system of equilibrium (§ 35) ; hence these values would necessarily make the algebraic sum of resolutes in the third direction zero, and therefore (§ 30, Cor. 2) would *necessarily* satisfy the third equation. Hence the third equation would not be independent of the other two equations.

Again, in Fig. 48*f*, we had *five* unknowns P, Q, R, S, T. Also by applying Lami to each of the *three* systems of

forces at **A**, **B**, **C** respectively, we could have obtained 3×2, *i.e. six* equations. This is one more than is necessary.

We shortened the work by applying the principle of symmetry, and thus assuming that R and T were equal to P and S respectively. This practically reduced the number of unknowns to three, viz, P, Q, S.

Summary of Results.

For a body on a *smooth inclined plane* under a *horizontal force*,—

$$P = W \tan i \quad \ldots\ldots (1). \qquad (\S\ 44.)$$

$$R = W \sec i \quad (2). \qquad (\S\ 44.)$$

$$P : R : W = h : l : b \quad \ldots\ldots (3). \qquad (\S\ 44.)$$

For a body on a *smooth inclined plane* under a force acting *parallel to the plane*,—

$$P = W \sin i \quad \ldots\ldots (4). \qquad (\S\ 45.)$$

$$R = W \cos i \quad \ldots\ldots (5). \qquad (\S\ 45.)$$

$$P : R : W = h : b : l \quad \ldots\ldots (6). \qquad (\S\ 45.)$$

To solve a problem involving *three forces* in equilibrium at a point apply

(i) *the converse of the Triangle of Forces* (§ 43.)

or (ii) *Lami's Theorem,* (§ 48.)

or (iii) *Resolution of Forces.* (§ 47.)

To solve a problem involving *more than three forces* in equilibrium at a point apply

Resolution of Forces. (§ 47*a*.)

From the *conditions of equilibrium* of a set of forces acting at a point we can derive *two and only two independent equations.* (§ 48*a*.)

In general the *number of independent simultaneous equations* required to solve a problem must be *equal to the number of unknown quantities* in the problem. (§ 48*a*.)

EXAMPLES III.

(*Selected Problems* :—1, 2, 4, 5, 7, 9, 10, 12, 14, 20, 22, 23, 25, 26, 27, 29, 30, 34, 35, 36, 38.)

1. Find (*a*) the horizontal forces, and (*b*) the forces up the plane required to support each of the following weights on the given inclined planes :—

(i.) 10 lb. on an incline of length 10 ft. and height 6 ft. ;
(ii.) 78 lb. on an incline of height 5 ft. and base 12 ft. ;
(iii.) 30 tons on an incline of length 25 yd. and base 24 yd. ;
(iv.) 85 kilog. on an incline of 8 in 17 of length 34 metres.

2. Find the horizontal force, and find also the force acting up the plane, required to support a weight of

(i.) 5 tons on an incline of 30° ;
(ii.) 28 lb. on an incline of 45° ;
(iii.) 10 kilog. on an incline of 60° ;

In each case find the reaction of the plane.

3. A weight of 9 lb. is pulled up an inclined plane of which the height is $1\frac{1}{2}$ ft. and the base is 2 ft. What force (i.) acting horizontally, (ii.) acting along the plane, is required for the purpose ?

4. A weight of 12 lb. is supported by two strings, each inclined at an angle of 45° to the vertical. Find the tension of the strings.

5. A weight of 10 lb. is supported by two strings which are inclined at angles of 30° to the vertical. Find the tensions in the strings.

6. A small heavy ring, which can slide freely upon a smooth thin rod, is attached to the end of the rod by a fine string. If the rod be held in any position inclined to the vertical, draw a triangle representing the forces acting upon the ring.

7. A weightless wire is stretched between two points **A, B,** on the same level 60 inches apart ; a weight of 5 lb. is hung at the middle of the wire, and depresses it one inch below the line **AB**. Find approximately the tension of the wire.

8. A wire 40 inches long is stretched with a force of 50 lb., find roughly the force which must be applied to the middle point to deflect it laterally half an inch out of the line joining the ends.

9. A heavy particle is supported by means of two strings attached to it. If the direction of one string be given, find that of the other in order that its tension may be the least possible (Use Lami's Theorem).

10. A body of weight 10 lb. rests on a smooth plane inclined at an angle of 30° to the horizontal. Find the least value of the force required to sustain it and the reaction of the plane.

11. A weight rests on a smooth inclined plane. Determine the direction and magnitude of the least force which will keep it in equilibrium. Find also the direction of the force in order that the thrust on the plane may be double of that exerted in the first case.

12. Two weights support each other on two smooth inclined planes of the same height, the weights being connected by a string passing over a smooth pulley at the junction of the planes. The angles of the planes are 30° and 60°. What is the ratio of the weights and the tension of the string?

13. A wedge, whose triangular section **ABC** is right-angled at **A**, is placed with **BC** on a horizontal plane; a horizontal force P will support a weight Q placed on **AC**?

14. A picture, weighing 56 lb., is slung over a nail in the ordinary way by a cord attached to two eyes in the top horizontal bar of its frame. If the height of the nail above this bar is half the distance between the eyes, what is the tension in the cord? Under what circumstances would the tension be equal to or greater than the whole weight of the picture?

15. A particle **P** is placed inside a smooth circular tube, and is acted on by two forces towards the extremities **A** and **B** of a fixed diameter **AB**. The forces are respectively proportional to **PA** and **PB**. Find the position of equilibrium.

16. A wedge, whose triangular section **ABC** is right-angled at **A**, is placed with **BC** on a horizontal plane. Two weights P and Q, connected by a string, will balance if placed with the string passing over **A** and P resting on **AB**, and Q on **AC**. If **AB** be placed horizontally, they will balance with P on **BC** and Q hanging vertically. Show that the sides of the triangle are in a geometrical progression whose common ratio is equal to the ratio $P : Q$.

17. Determine the tension of the threads of a rectangular piece of network hung from a horizontal bar, due to suspending a series of equal weights in a horizontal line at the lowest points of the net, supposing the meshes are equal hexagons of which a pair of sides are vertical.

18. **AB** and **AC** are two smooth planes inclined to the horizon at angles α and β respectively, and meeting at their highest point **A**. Two weights W and V, one on each slope, balance one another when connected by a string passing over a smooth pulley at **A**. If, when the inclination of each plane is doubled, equilibrium is maintained by replacing V by V', prove that

$$V \cos \alpha = V' \cos \beta.$$

19. A weight is supported on a smooth plane of inclination α, by a string inclined to the horizon at an angle γ. If the slope of the plane be increased to β, and the slope of the string unaltered, the tension of the string is doubled. Prove that

$$\cot \alpha - 2 \cot \beta = \tan \gamma.$$

20. Two weights W and $W\sqrt{3}$ are connected by a string which passes over a smooth pulley. W hangs freely, and $W\sqrt{3}$ rests on a smooth plane inclined at an angle of 30° to the horizon. In the position of equilibrium find the inclination to the vertical of the string between the pulley and the weight $W\sqrt{3}$.

21. **AB** and **AC** are two smooth planes inclined to the horizon at angles α and β respectively. D is a smooth pulley vertically above **A** over which passes a string attached to two weights W and $2W$ which rest on **AB** and **AC** respectively. The two portions of the string are inclined to **AB** and **AC** respectively at angles β and α respectively. Prove that

$$\sin 2\alpha = 2 \sin 2\beta.$$

22. **A** and **B** are two points in the same horizontal plane, 2 ft. apart. A weight of 30 lb. is to be supported at **C** by equal strings **CA**, **CB** attached to **A** and **B**. If neither string can bear a strain of more than 39 lb. wt., calculate the smallest length of string which may be used.

23. Two smooth planes inclined at angles α and β to the horizon, meet at their lowest point, and a sphere of weight W rests between them. Find the thrust on each plane.

24. A weight is supported on a smooth inclined plane by a string attached to the top of the plane. At three different inclinations of the plane the tensions are proportional to $\sqrt{3}$, $\sqrt{2}$, 1 respectively. Find the three inclinations given that they are in arithmetical progression.

25. **ABC** is an equilateral triangle formed of string. It is kept in equilibrium by three forces at **A**, **B** and **C** whose lines of action if produced backwards bisect the angles of the triangle.

(1) Prove that these three forces are of equal magnitude.

(2) Find the tension in each side of the triangle, if each outward force is P.

26. **A** and **B** are two smooth pulleys in the same horizontal plane, $2x$ feet apart; a string **DACBE** passing over the two pulleys supports weights W, W, and X at **D**, **E**, and **C** respectively. **C** rests p feet below **AB**. If a second weight X be attached at **C**, **C** rests q feet below **AB**. Find x in terms of p and q.

27. A weight of $\sqrt{3}$ lb. rests at a point **Q** on the surface of a smooth sphere of centre **O**, and radius r, being supported by a string **QP** of length r, whose other extremity is fastened to a point **P** vertically above **O**. If $PO = r\sqrt{3}$, find the tension of the string.

28. Find the pressure on a smooth plane inclined at an angle α to the horizontal, when a weight W is supported on it by a force whose line of action makes an angle β with the horizontal.

If the pressure on the plane be the arithmetic mean between the weight and the force, and if $\beta = 2\alpha$, find $\sin \beta$.

29. A weight $2P$ is kept in equilibrium on an inclined plane by a horizontal force P, and a force P acting parallel to the plane. Find the ratio of the length, base, and height of the plane, and the pressure on the plane.

30. A body of weight W is supported on a smooth inclined plane, by a force P inclined to the plane, and the pressure on the plane is R.

Given that
$$P = \frac{1}{\sqrt{3}} W$$
and
$$R = \frac{1}{\sqrt{3}} W$$
find the inclination of the plane, and the direction of P.

31. What would be the answer to the last question if
$$P = \tfrac{1}{2}\sqrt{2}W$$
$$R = \tfrac{1}{2}(\sqrt{3} - 1)W$$

32. A bead can slide on a smooth vertical circular wire, and is connected by a string (equal in length to $\sqrt{3} \times$ radius) to the highest point of the circle.

Find (i) the tension of the string.
(ii) the pressure of the bead on the wire.

33. Three equal rings rest on a smooth vertical circular wire at the corners of an equilateral triangle of which one side is vertical, the uppermost being connected with the other two by means of strings. Find their tensions.

34. An endless string **ABCDA** passes over two pulleys at **B** and **D** which are on the same level and two ft. apart; weights are attached at **C** and **A**: and, in equilibrium, the system is symmetrical about the vertical line **AC**.

Given that **C** and **A** are respectively 5 and 16 in. below **BD**, find the ratio of the weights.

35. A mass of 10 lb. rests on a smooth plane inclined at 30° to the horizontal, and is attached to a string which passes over a smooth pulley at the top of the plane and supports a mass of 2 lb. hanging freely. What horizontal force applied to the 10 lb. mass will maintain equilibrium?

36. A mass of 10 lb. is supported by three strings which lie in one vertical plane. Two of these strings are inclined to the upward vertical at 60°, one on each side of the vertical. The third is inclined to the upward vertical at 30°, and passing over a smooth pulley supports a mass of $2\sqrt{3}$ lb. Find the tensions of the other two strings.

37. **XOY** is a straight line. Four forces P, Q, R, S acting on **O** are in equilibrium. P and Q makes angles α with **OX**, one on each side R and S makes angles β with **OY**, one on each side. P and S are on the same side of **XOY**. Prove that

$$R \sin 2\beta = P \sin (\alpha + \beta) - Q \sin (\alpha - \beta).$$

38. Four forces acting at a point are in equilibrium. Their magnitudes are $1, \sqrt{3}, 2, 2$. If the first two forces are at right angles, in what directions do the other two forces act?

CHAPTER IV.

THE TRANSMISSION OF FORCE.

49. **Rigid bodies.**—In treating the conditions of equilibrium of several forces acting "*at a point*," we have supposed the forces to be all applied to a single particle placed at that point. When two or more forces act *in parallel straight lines*, it is impossible to suppose them to be applied to the same particle, for parallel lines never meet. They must therefore, be supposed to be applied to a body of extended size. Accordingly, it will be necessary to state what is meant by a rigid body before proceeding further.

DEFINITION.—*A* **rigid body** *is a body whose size and shape always remain the same whatever forces be applied to different parts of it.*

By this it is implied that *the distance between any two particles of a rigid body always remains the same.*

In reality no body is perfectly rigid, but many solid bodies may be regarded as rigid for all practical purposes.

50. **Rotation.**—A rigid body can be **rotated.**

When a top is spinning, the different particles of the top *rotate* or move round and round rapidly, though the top does not change its position as a whole. The same is true when a wheel *rotates* or turns round. When a door is opened, the hinge remains where it is and the other parts of the door *rotate* or turn about it, those furthest from the hinge moving most rapidly.

When a body changes its position as a whole, this motion is said to be a motion of *translation* to distinguish it from rotation.

When a number of forces acting on a rigid body are in equilibrium, they must have no tendency to produce motion *either of translation or of rotation*; otherwise the body will not remain at rest.

51. **Statical Property of a Rigid Body.**—In the first place, it will be observed that

**Two forces acting at two points of a rigid body are in equilibrium if, and only if, they are equal and opposite, and act in the same straight line.*

This may readily be verified by attaching strings to two points **A, B** of a body (say a board) resting on a horizontal table, and pulling their ends apart horizontally. The body will turn round until the

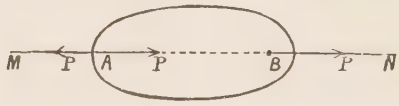


Fig. 49.

strings **MA, BN** are both in one straight line, and will then come to rest. And, since the forces produce no motion of the body as a whole (*i.e.* no motion of translation), they must be equal and opposite.

[If the body is again displaced so as to bring the strings out of one straight line, it will not remain at rest, but will rotate back to its former position. Hence two equal and opposite forces which do *not* act in the same straight line are not in equilibrium, but tend to produce rotation.]

From the above property we deduce the following principle :—

52. **Principle of Transmission (or Transmissibility) of Force :**

A force acting on a rigid body may have its point of application transferred to any point whatever in the straight line in which it acts without affecting the conditions of equilibrium.

Let P be any force applied to a body at **B** in the direction **BN**. Let **A** be any point of the body in the straight line **BN** or **BN** produced. At **A** apply two equal and opposite forces of magnitude P in the straight line **AB**. These two forces balance each other, and therefore do not affect the conditions of equilibrium of the original forces. Now consider the two forces $+P$ at **B** and $-P$ at **A**. By the

* A strict proof of this is given in § 217*a*.

property just proved, these two forces balance each other, and therefore they can be removed without affecting the conditions of equilibrium. We are, therefore, left with the force $+P$ at **A** as the statical equivalent of the original force P at **B** applied in the same straight line; as was to be proved.

The principle of the Transmission of Force may also be stated thus:

When a force acts on a rigid body, it is immaterial what point in its line of action is considered to be the point of application of the force.

The point of application may even be taken *outside the body*, provided that the force is applied to a particle rigidly connected with the body. But a force *cannot* be replaced by an equal and parallel force acting at any point *not* in its original line of action.

Tension of a string.—In the same way it is evident that *the tension of a stretched string is the same at all points of its length.* This property is very important.

53. Conditions of equilibrium of three forces in one plane.

If a rigid body is in equilibrium under three forces in one plane, their lines of action must all be parallel or all pass through one common point.

For let the three forces be not all parallel. Then the lines of action of two of them must meet in some point, say **O**. By the principle of Transmission of Force, we may suppose that two forces to be applied to a particle of the body (or rigidly connected with the body) at **O**. Hence, as in Chap. I., they are equivalent to a single resultant force acting at **O**. This resultant and the third force balance; therefore they must be equal and opposite and in the same straight line (§ 51). Hence the line of action of

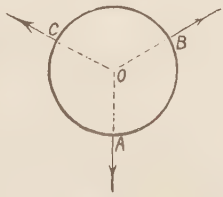


Fig. 50.

the third force must pass through **O**, and therefore the three lines of action must all pass through one common point.

In addition to passing through one common point, the three forces must satisfy the same conditions of equilibrium as if they acted on a particle at that point ; that is, they must be capable of being represented in magnitude and direction by the sides of a triangle taken in order.

The conditions of equilibrium of three *parallel* forces will be investigated in Chap. VI.

54. *The point of intersection of the forces need not be within the body.*

Thus, let three cords be attached to a ring or hoop at the equidistant points **A, B, C** (Fig. 50), and let these cords be pulled with equal forces in the direction of radii **OA, OB, OC.** Then these forces will be in equilibrium, for their directions pass through one common point (viz., the centre **O**), and are inclined at angles of 120°. Hence the ring will remain at rest notwithstanding the fact that the point **O** is not in the substance of the ring itself. (See also § 58, Ex. 1.)

55. **The wedge** is a triangular block which is used either for splitting a body (*e.g.*, a piece of wood) into two parts, for separating two bodies, or for slightly raising heavy weights off the ground. The section of the block is a triangle, and the wedge generally studied on account of its greater utility and simplicity is isosceles, and could be formed by two right-angled inclined planes put back to back. A knife and a chisel afford good illustrations of the principle of the wedge.

The wedge we consider in theoretical mechanics is smooth ; but most wedges are so rough that, when they have been driven in between two bodies, the friction prevents them from coming out again. Usually, too, wedges are driven in by blows of a hammer ; hence the conditions of equilibrium found below are usually far from being realised in practice.

56. **Conditions of equilibrium of a smooth wedge.**—Let a smooth wedge, whose section is the triangle **ABC,** be driven in between two bodies **M, N** by a force P applied on its face

BC. Let Q, R be the reactions with which **M, N** resist the insertion of the wedge.

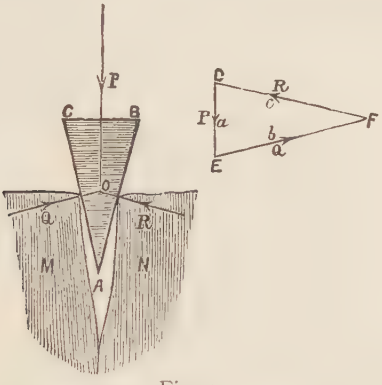


Fig. 51.

Then the wedge is kept in equilibrium by the three forces, P, Q, R, and, since the wedge is smooth, these forces act perpendicularly to **BC, CA, AB,** their directions meeting in some point **O.**

Therefore, by the Perpendicular Triangle of Forces (or as shown by the triangle **DEF**), P, Q, R can be represented in magnitude by **BC, CA, AB,** respectively, that is,

$$\frac{P}{\mathbf{BC}} = \frac{Q}{\mathbf{CA}} = \frac{R}{\mathbf{AB}};$$

or, if a, b, c denote the lengths of **BC, CA, AB,**

$$\frac{P}{a} = \frac{Q}{b} = \frac{R}{c} \quad \ldots\ldots\ldots\ldots\ldots \quad (1).$$

In the case generally considered the section of a wedge is isosceles, or $b = c$. Therefore

$$Q = R = P \times \frac{b}{a}.$$

By sufficiently increasing the length b and making the breadth a very small, a very small force P can be made to overcome a very large resistance Q. A hatchet is a good example of this.

Example.—A stone rests against a vertical wall **AC**, and can be separated from the wall by a smooth wedge weighing 28 lb., whose triangular section **ACB** is right-angled at **C**, and has its sides 7, 24, 25 inches long. To find the horizontal force with which the stone presses against the wedge, no other forces being applied to the wedge.

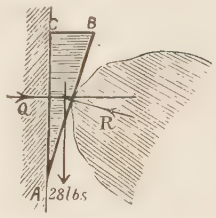


Fig. 52.

Let Q be the reaction of the wall, R the force of pressure of the stone on the wedge, P(= 28 lb.) the weight of the wedge. Then the wedge is kept in equilibrium by forces P, Q, R.

Since **ACB** is a right angle, **CB** is horizontal, and therefore P acts perpendicular to **BC**. Since the wedge is smooth, Q, R act perpendicular to **CA**, **AB** respectively. Therefore, by the Triangle of Forces,

$$\frac{P}{\mathbf{BC}} = \frac{Q}{\mathbf{CA}} = \frac{R}{\mathbf{AB}}, \text{ or } \frac{28}{7} = \frac{Q}{24} = \frac{R}{25};$$

whence $Q = 96$ lb., $R = 100$ lb.

Also, required horizontal component of R $= Q = 96$ lb.;
therefore the stone presses with a horizontal force of 96 lb.

57. **Equilibrium of a heavy body.—Applications to problems.**—The theorem of § 53 is very useful in enabling us to find the conditions of equilibrium of a heavy body supported at two given points by forces that are not vertical. The cases where the supporting forces are vertical will be considered later.

It will be proved in Chap. X. that the whole weight of a rigid body may always be supposed to act vertically at a single point of the body, called its *centre of gravity* or *centre of mass*. For the present, the following particular results will be assumed :—

(1) The weight of a heavy *uniform rod* or *beam* acts at its middle point.

(2) The weight of a *uniform sphere* or *cube*, or of a *circular disc*, acts at its centre.

It will also be necessary to remember (§§ 40, 42) that—

Action and reaction are equal and opposite (Newton's Third Law) ; and that

The reaction of a perfectly smooth surface is always perpendicular to that surface.

58. *Examples.*—(1) **Equilibrium of a ladder.**—A uniform ladder of weight W leans against a perfectly smooth wall. To find the thrusts which it exerts against the wall and ground when the ladder is 20 ft. long and reaches a height of 16 ft.

Let **AB** be the ladder, **G** its middle point.

Let P denote the reaction of the wall, R that of the ground.

Then the three forces acting on the ladder are—

(i.) Its weight W acting vertically through **G** (since the ladder is uniform);

(ii.) The reaction P at **B** acting horizontally (since the wall is smooth and vertical);

(iii.) The reaction R acting at **A**.

Since these forces are in equilibrium, they must pass through one point. Let the vertical through **G** meet the horizontal through **B** in **M**. Then the reaction **R** must act in the line **AM**.

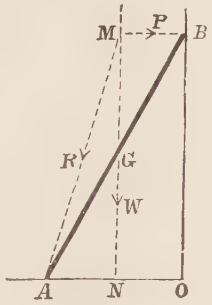


Fig. 53.

Let **MG** meet the ground in **N**. Then the forces W, P, R act in the directions of **MN, NA, AM**, the sides of the triangle **AMN** taken in order. Therefore, by the Triangle of Forces, these sides may be made to represent the forces in magnitude, so that if **BO** or **MN** represents W, **NA** and **AM** will represent P and R respectively. Therefore

$$P = W \times \frac{\mathbf{NA}}{\mathbf{MN}}, \qquad R = W \times \frac{\mathbf{AM}}{\mathbf{MN}}.$$

Now $\mathbf{AB} = 20$ ft., $\mathbf{BO} = 16$ ft.;

$$\therefore \quad \mathbf{AO}^2 = 20^2 - 16^2 = 4^2(5^2 - 4^2) = 4^2 \cdot 3^2 = 12^2;$$

$$\therefore \quad \mathbf{AO} = 12 \text{ ft.};$$

and it is easily seen* that

$$\mathbf{AN} = \tfrac{1}{2}\mathbf{AO} = 6 \text{ ft.}$$

Therefore also

$$\mathbf{AM}^2 = \mathbf{AN}^2 + \mathbf{NM}^2 = 6^2 + 16^2 = 2^2(3^2 + 8^2) = 2^4 \times 73;$$

$$\therefore \quad \mathbf{AM} = 2\sqrt{73} \text{ ft.}$$

$$\therefore \quad \text{Reaction of wall } P = W \times \frac{6}{16} = \frac{3W}{8},$$

$$\text{Reaction of ground } R = W \times \frac{2\sqrt{73}}{16} = \frac{W\sqrt{73}}{8}.$$

---

* For **AG** = **GB**; ∴ triangles **AGN, BGM** are equal in all respects; ∴ **AN** = **BM** = **NO** (the opposite side of the parallelogram **MBON**); ∴ **AN** = $\frac{1}{2}$**AO**.

(2) A heavy uniform rod of weight W is hinged to a point in a vertical wall, and is supported by a string attached to a point in the wall vertically above the hinge. The rod makes an angle α with the upward vertical, and an angle β with the string. Find the tension of the string.

Using Fig. 54, the rod **AB** is in equilibrium under three forces, viz. R the reaction of the hinge, T the tension of the string **BC**, W the weight of the rod acting vertically down from G the middle point of **AB**.

Let the vertical through **G** meet **BC** at **M**. Then *the lines of action of the three forces mus tbe concurrent*, therefore **AM** must be the line of action of R. Since **G** bisects **BA**, and **GM** is parallel to **AC**, therefore **M** bisects **BC**.

The sides of the triangle **AMC** will be proportional to the forces, for

R acts along **AM**,
T acts along **MC**,
W acts parallel to **CA**.

Thus $$\frac{T}{W} = \frac{\mathbf{MC}}{\mathbf{CA}} = \frac{\frac{1}{2}\mathbf{BC}}{\mathbf{CA}}.$$

But the sides of the triangle **ABC** are respectively proportional to the sines of the opposite angles.

Thus $$\frac{\mathbf{BC}}{\mathbf{CA}} = \frac{\sin \mathbf{BAC}}{\sin \mathbf{CBA}} = \frac{\sin \alpha}{\sin \beta}.$$

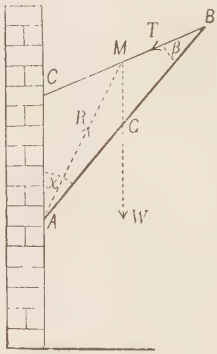


Fig. 54.

Hence $$\frac{T}{W} = \frac{1}{2} \cdot \frac{\sin \alpha}{\sin \beta};$$

$$\therefore \; T = \tfrac{1}{2} W \sin \alpha \, . \, \operatorname{cosec} \beta.$$

NOTE.—In each of the examples of this section we are using the Force Triangle method of solution; it is of course merely a question of convenience whether we use Force Triangles, or the Resolution of Forces, or Lami's Theorem.

(3) A ball 1 ft. in radius, and weighing 5 lb., rests against a smooth wall, and is attached to a string which passes through a hole in the wall at **A**, and is pulled with a force of 10 lb. To find the length of string projecting from the hole, and the reaction of the wall.

Let R be the reaction of the wall.
The forces on the ball are—

(i.) Its weight 5 lb. acting vertically through its centre **C**;

(ii.) The reaction R acting perpendicular to the wall, and therefore in direction **BC**;

(iii.) The pull of the string or 10 lb.

Since these pass through a point, the direction of the string passes through **C**, and the three forces are represented in direction by **AB**, **BC**, **CA**.

Therefore, if the length **AB** represents the weight 5 lb., **BC** will represent the reaction R and **CA** the pull of 10 lb.

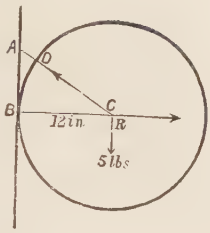


Fig. 55.

Now $\mathbf{CBA} = 90°$ and $\mathbf{AC} = 2\mathbf{AB}$;

$\therefore \angle \mathbf{BCA} = 30°, \quad \mathbf{BAC} = 60°$;

$\therefore \mathbf{BC} = \mathbf{AB}\sqrt{3}$;

whence $R = 5\sqrt{3}$ lb. $= 8{\cdot}66..$ lb.

But $\mathbf{BC} = 12$ inches;

$$\therefore \quad \mathbf{AC} = \frac{2}{\sqrt{3}}\mathbf{BC} = \frac{24}{\sqrt{3}} \text{ inches} = 8\sqrt{3} \text{ inches};$$

and, if **D** is the point of attachment of the string,

$$\mathbf{AD} = \mathbf{AC} - \mathbf{DC} = 8\sqrt{3} - 12 = 1{\cdot}856 \text{ in.}$$

Therefore the required reaction is $8{\cdot}66..$ lb. and $1{\cdot}856$ in. of the string projects from the wall.

59. **Theorems.**—(1) Three forces proportional in magnitude to the sides of a triangle and bisecting these sides at right angles will be in equilibrium if they act all inwards or all outwards.

For, by Euc. IV. 5, the perpendicular bisectors of the sides all meet in the centre of the circumscribing circle. Hence the forces all pass through one point.

Moreover, if the triangle is turned through a right angle, its sides, taken in order, will be parallel to the forces, and will represent them both in magnitude and direction. Hence the forces are in equilibrium.

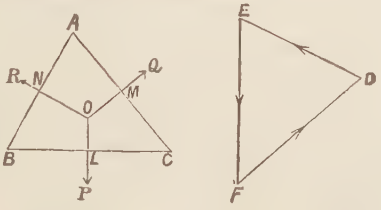


Fig. 56.

(2) Any number of forces proportional to the sides of any polygon and bisecting these sides at right angles will be in equilibrium if they act all inwards or all outwards.

Let **ABCDE..** be the given polygon. Join **AC, AD**..., and suppose the forces to all act outwards.

From the last theorem, the forces of magnitude **AB, BC**, bisecting **AB, BC** at right angles, have a resultant of magnitude **AC**, bisecting **AC** at right angles, acting *inwards* with respect to the triangle **ABC**, or *outwards* with respect to **ACD**.

Similarly, the forces **AC** and **CD** have a resultant **AD** bisecting **AD** at right angles.

Finally, the forces **AD, DE, EA**, bisecting their respective sides at right angles, are in equilibrium.

Therefore the given forces are in equilibrium. The same is similarly true if they act inwards.

(3) To deduce the conditions of equilibrium of three parallel forces acting on a rigid body.*

Three forces P, Q, R acting perpendicular to the sides of a triangle **ABC** at the angular points will meet in the orthocentre, and therefore (as in Theorem 1) they balance if they are proportional to the sides, that is, if

$$\frac{P}{BC} = \frac{Q}{CA} = \frac{R}{AB}.$$

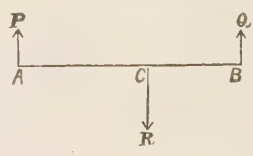


Fig. 57.

This is true however small the altitude of the triangle **ABC**. Hence it must be true when the altitude is infinitesimal. In this case the points **A, B, C** lie in one straight line, and the forces P, Q, R act perpendicular to this line, and are therefore parallel.

Hence the conditions of equilibrium of the three parallel forces P, Q, R acting at **A, B, C** become

$$\frac{P}{BC} = \frac{Q}{AC} = \frac{R}{AB} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

In the above figure $BA = BC + CA$,
and **AB** is equal and opposite to **BA**.

Therefore, from (1), R is equal and opposite to $P + Q$, so that, if the directions of forces are denoted by their signs, then, *algebraically*,

$$P + Q + R = 0$$

is a necessary condition of equilibrium.

---

* The conditions of equilibrium of three or more parallel forces will be treated more fully by other methods in Chap. VI. The present treatment may therefore be omitted on first reading, although it affords an instructive exercise.

59*a*. **On Projections.—The projection of a given point on a given straight line is the foot of the perpendicular drawn from the point to the line.** Thus in Fig. 57*a* the projection of the point **B** on the line **XY** is the point *b*.

The projection of a line on a given straight line is that portion of the given straight line which lies between the projections of the two extremities of the given line. Thus in Fig. 57*a* the projection of the line **AB** on the straight line **XY** is the line *ab*.

The first of the two lines (the line which is projected) may be either *straight*, or *curved*, or "*broken*" (*i.e.* consisting of a series of straight or curved lines joined together). The second of the two lines (on to which the first is projected) *must however be a straight line*, since the definition involves perpendiculars to this line. Thus in Fig. 57*a* the projection of the curved line **CD** on the straight line **XY** is *cd*; and in Fig. 57*b* the projection of the broken line **EFGHK** on the straight line **PQ** is *ek*.

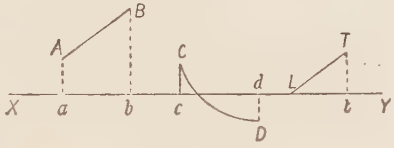


Fig. 57*a*.

Note that in Fig. 57*a* the projection of the point **L** is **L** itself, and the projection of the line **LT** is **L***t*.

In projections we generally adopt the usual convention of signs, viz. that when the letters of the projection are quoted *in one order* (usually from left to right) the projection is reckoned *positive*, and when quoted in the reverse order the projection is reckoned negative. Thus in Fig. 57*a* the projection of the line **CD** is *cd* which is reckoned positive, while the projection of the line **DC** is *dc* which is reckoned negative.

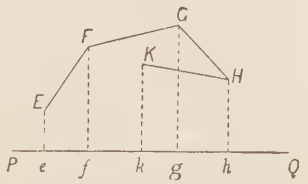


Fig. 57*b*.

Adopting this convention it is easy to see that **the projection of a broken line is equal to the algebraic sum of the projections of its component parts.** For example, in Fig. 57*b*, the projection of the broken line **EFGHK** is *ek* (by definition). Also the algebraic sum of the projections of its component parts **EF**, **FG**, **GH**, **HK** will be

i.e. $ef + fg + gh + hk$

$ef + fg + gh + kh$

which is again equal to *ek*.

To find the algebraic projection of one straight line on another.

In Fig. 57*c* it is required to find the projection of **OH** on **PQ**.

Draw **OX** *parallel to the positive direction of* **PQ**.

Then projection of

$$\mathbf{OH} = oh = \mathbf{OM} = \mathbf{OH} \cdot \frac{\mathbf{OM}}{\mathbf{OH}}$$

$$= \mathbf{OH} \cdot \cos \mathbf{HOX}.$$

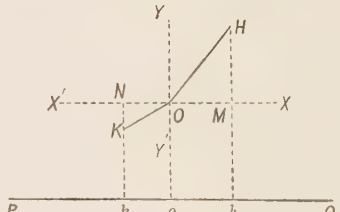


Fig. 57*c*.

Thus **to find the projection of OH we multiply it by the cosine of the angle which OH makes with OX**, *i.e.* with the positive direction of **PQ**.

The angle **HOX** is positive and acute, but it will be found that the above rule holds, whatever the inclination of the projected line may be.

Thus if it be required in Fig. 57*c* to find the projection of the line **OK** :—

In whatever position **OK** may fall, we may use the triangle **KON** as the triangle of reference to determine the ratios of the angle **KOX**. Thus $\cos \mathbf{KOX} = \frac{\mathbf{ON}}{\mathbf{OK}}$.

Hence, adopting throughout the usual conventions of sign, we have for all positions of **OK**

$$\text{projection of } \mathbf{OK} = ok = \mathbf{ON} = \mathbf{OK} \cdot \frac{\mathbf{ON}}{\mathbf{OK}}$$

$$= \mathbf{OK} \cdot \cos \mathbf{KOX}.$$

Corollary.—From § 32, or otherwise, it is now obvious that if a force represented by **OK** be resolved in a direction parallel to **PQ**, the projection *ok* will represent the resolute in magnitude and direction.

In a similar way it can be proved that the projection of **OK** on a line perpendicular to **PQ** is **OK** sin **KOX**, upward lines being reckoned positive.

59*b*. The following example will show the use of the theory of projections in mechanical problems. § 59*a* determines both the sign and magnitude of the projection of a line ; but it is frequently more convenient to determine the *sign* of the projection by inspection, and its *magnitude* by multiplying by the cosine of the *acute* angle between the projected line and the line on to which it is projected. The latter method has been used in this case.

Example.—A uniform square lamina* rests in equilibrium in a vertical plane under gravity with two of its sides in contact with smooth pegs in the same horizontal line at a distance c apart. Show that the angle θ made by a side of the square with the horizontal in a non-symmetrical position of equilibrium is given by the equation $c(\sin\theta + \cos\theta) = a$, $2a$ being the length of a side of the square.

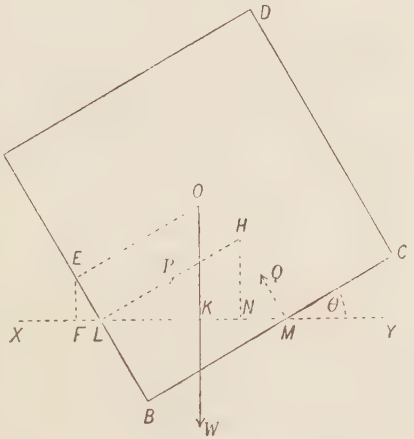


Fig. 57*d*.

ABCD represents the square (Fig. 57*d*) ; **O** is its centre, at which the whole weight of the square may be supposed to act. Then if **OE** is drawn parallel to **BC**, **OE** = **EB** = half the side of the square = a.

* "Lamina" denotes a flat sheet of material assumed to be without sensible thickness.

Let **L** and **M** be the two pegs ; then **LM** is horizontal and equal to c, and the angle **CMY** is θ. Then since the pegs are smooth, their reactions P and Q act at right angles to **AB, BC** respectively. Thus if the lines of action of these reactions intersect at **H**, **LHM** is a right angle, and $\mathbf{HLM} = \theta$.

$$\therefore \mathbf{HL} = \mathbf{LM} \cos \mathbf{HLM} = c \cos \theta,$$

and $$\mathbf{BL} = \mathbf{HM} = \mathbf{LM} \sin \mathbf{HLM} = c \sin \theta.$$

Draw **EF, OK, HN,** perpendicular to **XY,**

Then **LN** = projection of **LH** on **XY** = $\mathbf{LH} \cos \theta = c \cos^2 \theta$.

FK = projection of **OE** on **XY** ; but **EO** is parallel to **BC** and therefore is inclined at an angle θ to **X Y** ; thus $\mathbf{FK} = \mathbf{EO} \cos \theta = a \cos \theta$.

FL = projection of **EL** on **XY** ; but **EL** is perpendicular to **BC** and therefore is inclined to **XY** at an angle $90° - \theta$* ; thus

$$\mathbf{FL} = \mathbf{EL} \cos (90° - \theta) = (\mathbf{EB} - \mathbf{BL}) \sin \theta = (a - c \sin \theta) \sin \theta$$
$$= a \sin \theta - c \sin^2 \theta.$$

Now the square is to be in equilibrium under the forces P, Q, and W ; hence the lines of action of these forces must be concurrent ; this is not so in Fig. 57*d*, which therefore does *not* represent an equilibrium position.

For equilibrium it is necessary that **OW** should pass through **H**, and therefore that **O** and **H** should lie in the same vertical line. Hence **K** and **N** (the projections of **O** and **H** on **XY**) must coincide, and therefore **LK** = **LN**,

i.e. $$\mathbf{FK} - \mathbf{FL} = \mathbf{LN} ;$$

whence, using the preceding results,

$$a \cos \theta - (a \sin \theta - c \sin^2 \theta) = c \cos^2 \theta,$$
$$\therefore a (\cos \theta - \sin \theta) = c (\cos^2 \theta - \sin^2 \theta),$$
$$\therefore a = c (\cos \theta + \sin \theta).$$

Summary of Results.

By the *Principle of Transmission of Force*, the effects of a force on a rigid body is the same at whatever point in its own line of action the force may be applied. (§ 52.)

The *lines of action* of three forces which are in equilibrium must either be *parallel* or *concurrent*. (§ 53.)

Conditions of equilibrium of a *smooth wedge* under forces P, Q, R perpendicular to faces a, b, c are

$$P/a = Q/b = R/c \ldots\ldots\ldots (1) \qquad (\S\ 56.)$$

* $\angle \mathbf{ELF} = \mathbf{BLM} = 90° - \mathbf{LMB} = 90° - \theta$.

The weight of a uniform *rod* acts at its *middle point.* (§ 57.)

The weight of a uniform *sphere, cube, circular lamina, or ring* acts at its *centre.* (§ 57.)

To solve a problem involving *three non-parallel forces* in equilibrium *applied at different points* use the condition of *concurrency* and also the *Triangle of Forces,* or *Lami's Theorem,* or *Resolution of Forces.* (§ 58.)

The *projection* of a " broken line " is equal to the *algebraic sum* of the projections of its parts. (§ 59*a*.)

The *projection* of a line of *length* r inclined at an *angle* θ to the *positive* direction of the base line is $r \cos \theta$. (§ 95*a*.)

EXAMPLES IV.

(*Selected Problems* .—1, 2, 5, 8, 9, 10, 11, 13, 14, 15, 18, 20, 22, 23, 25, 27, 29, 30.)

1. A person ascends a ladder resting on a rough horizontal floor against a smooth vertical wall; determine the direction and magnitude of the force with which the ladder presses against the floor.

2. A uniform rod **BC**, 6 in. long, weighs 2 lb., and can turn freely about its end **B**. It is supported by a string **AC**, 8 in. in length attached to a point **A** in the same horizontal line as **B**, the distance **AB** being 10 in. Find, by a diagram, the tension of **AC**.

3. **AB** is a wall, and **C** a fixed point at a given perpendicular distance from it; a uniform rod of given length is placed on **C**, with one end against **AB**. If all the surfaces are smooth, find the position in which the rod is in equilibrium.

4. Two forces applied at two points **A, B** of a rigid body in the straight line **AB** are such as to balance one another. Prove that, when the body moves in the direction **AB**, the works done by the forces are equal and opposite.

5. A uniform rod, whose length is 8 ft. and weight 16 lb., is placed over a smooth peg, so that one end rests against a smooth vertical wall. The distance of the peg from the wall is 6 in. Find the position of equilibrium and the force of pressure on the peg.

6. A uniform rod, whose length is 2 ft. and weight 1 lb., is placed over a smooth peg, so that one end rests against a smooth vertical wall. The thrust on the peg is 2 lb. Find the distance of the peg from the wall and the position of equilibrium.

7. A uniform rod **AB**, inclined at an angle of 30° to the horizon, rests with the end **A** in contact with a rough horizontal table, the end **B** being supported by a string attached to a point **C** vertically above **A**. If **BC** be inclined at an angle of 60° to the horizontal, find the reaction of the table and the tension of the string.

8. A rod **AB** is hinged at **A**, and supported in a horizontal position by a string **BC** making an angle of 45° with the rod. The rod has a weight of 10 lb. suspended from **B**. Find the tension in the string and the force at the hinge. The weight of the rod may be neglected.

9. A rectangular box, containing a ball of weight W, stands on a horizontal table, and is tilted about one of its lower edges through an angle of 30°. Find the thrusts between the ball and the box.

10. **AC** and **AB** are two smooth inclined planes at right angles to one another and intersecting at their lowest point **C**. A uniform heavy rod **AB** rests in equilibrium against them. Show that its middle point is vertically above **C**.

11. Construct a triangle whose sides represent the forces acting on the rod in Q. 10, and calculate the forces of pressure of the rod against the planes, the inclinations of the planes being 30° and 60°, and the weight of the rod 1 lb.

12. A uniform rod rests with its lower end in contact with a smooth vertical wall. It is supported by a string of length a, one end of which is fastened to a point in the wall, and the other end to a point in the rod at a distance a from its lower extremity.

Find the length of the rod; and show that if the rod is of the proper length it can rest at any inclination to the vertical.

13. A bracket **ABC**, right-angled at **B**, and having **AB** horizontal, is smoothly hinged to a wall at **B** and **C**, so that **BC** is vertical, and the action at the hinge **B** is horizontal; it supports a weight W at **A**. Calculate the stresses at the hinges **B** and **C**, supposing the weight of the bracket to be neglected; given $\mathbf{AB} = p$, $\mathbf{BC} = q$.

14. A sphere of radius r rests against a smooth vertical wall, to which it is attached by a string of length l fastened to a point on the surface. Find the tension of the string.

15. A sphere of radius r, and weight W, rests on a smooth plane (inclined at an angle of 45° to the horizontal), to which it is attached by a string of length r fastened to a point on its surface. Find the tension of the string.

16. Two smooth spheres, each of weight 3 lb. and radius 5 inches, are suspended from a point to which each is attached by a string 8 in. long.

Assuming that they hang symmetrically in contact, find the stress at the point of contact.

17. In Example (2), § 58, show that if θ be the inclination of the reaction at the hinge to the vertical,

$$\tan\theta = 2\sin\alpha \,.\, \sin(\alpha+\beta)/[3\sin\beta + \sin(2\alpha+\beta)].$$

18. One end of a uniform rod is attached to a hinge, and the other end is supported by a string, attached to the extremity of the rod and inclined to the horizontal at the same angle θ as the rod.

If W be the weight of the rod, prove that the strain on the hinge is

$$\tfrac{1}{4}W\sqrt{8+\operatorname{cosec}^2\theta}.$$

19. A heavy uniform rod **AB** is to be supported in a given position, with its upper end **A** resting at a given point against a smooth vertical wall, by means of a fine string attached to its lower end **B**, and to a point **C** in the wall.

Find by a geometrical construction the point in the wall to which the string must be attached.

20. It is required to place a heavy uniform rod in equilibrium with one end resting against a smooth vertical wall, and the other against a smooth plane inclined to the wall at an angle θ. Prove that the inclination of the rod to the horizontal is

$$\tan^{-1}(\tfrac{1}{2}\tan\theta).$$

21. It is required to place a heavy uniform rod in equilibrium with its ends resting one on each of two smooth planes, which meet at their lowest points, and are inclined at angles α and β to the horizontal. Prove that the inclination of the rod to the horizontal is

$$\tan^{-1}(\tfrac{1}{2}\cot\beta - \tfrac{1}{2}\cot\alpha).$$

22. One corner of a uniform square lamina is fastened by a string to a point in a smooth vertical wall; an adjacent corner rests against the wall, and the plane of the lamina is vertical. If the length of the string is equal to that of one side of the square, find the inclination of the string to the vertical.

23. A smooth hemispherical bowl of diameter d is fixed with its rim in a horizontal plane. A uniform rod of length l rests with its lower end against the inner surface of the bowl, and its upper end projecting over the rim. Show that if θ be the inclination of the rod to the horizontal

$$2d \cos 2\theta = l \cos \theta.$$

24. A uniform rod of length $2l$ rests with its lower end in contact with a smooth wall. It is supported by a string of length a, one end of which is fastened to a point in the wall, and the other end to a point in the rod at a distance b from its lower extremity. If the inclination of the string to the vertical be θ show that

$$\cos^2\theta = \frac{b^2(a^2 - b^2)}{a^2 l(2b - l)}.$$

25. A rectangular board **ABCD** is hinged at **A** to a wall, so that its plane is vertical. It is supported by a smooth peg **P**, which is in contact with the side **BC**. The reaction at **A** acts along **AD**,

Prove that

$$2c = b + a \tan \theta,$$

where $\mathbf{AB} = a$, $\mathbf{BC} = b$, $\mathbf{PB} = c$, and θ is the inclination of **AB** to the downward vertical.

26. A heavy uniform rod **AB**, of length $2l$ and weight W, rests on a smooth peg at **C**, and is inclined at an angle θ to the horizontal. It is kept in equilibrium by a string attached to its upper end **A**, which passes over a pulley **D**, and supports a weight Q. The horizontal distance between **C** and **D** is a, and the vertical distance b, and **D** is on the same level as **A**. Prove that the value of θ is derived from the equation

$$l \sin^3\theta - l \sin \theta + b = 0,$$

and that

$$Q = W \tan \mathbf{O}.$$

27. A smooth sphere, of weight W and radius r, is resting between two parallel horizontal bars. The horizontal distance between the bars is a and the vertical distance is b ; find the pressure on the lower bar.

28. A rectangular board **ABCD** is hinged at **A**, so that its plane is vertical, and is supported by a horizontal string attached to the highest point **D**.

Given AD $= a$, AB $= b$, and that AD makes an angle α with the upward vertical, prove that the reaction at A makes with the upward vertical an angle

$$\tan^{-1}\tfrac{1}{2}\left(\tan\alpha + \frac{b}{a}\right).$$

29. A sphere of radius r rests on a smooth plane inclined to the horizontal at an angle α, to which it is fastened by a string attached to its surface.

Find the length of the string given that its tension is equal to the weight of the sphere.

30. In the preceding problem, find the length of the string given that its tension is equal to the pressure on the plane and prove that this is only possible when $\alpha > 45°$.

31. A string of length $2l$ has one end attached to the extremity of a smooth uniform heavy rod of length $2a$, and the other carries a weightless ring which slides on the rod. The rod is suspended by means of the string from a smooth peg. Prove that if θ be the angle which the rod makes with the vertical, then

$$l\cos\theta = a\sin^3\theta.$$

EXAMINATION PAPER II.

1. State and prove the principle of Transmissibility of Force.

2. If three forces in one plane keep a body in equilibrium, show that their lines of action will either meet in a point or be all parallel.

3. If a body is partly supported by resting against a smooth fixed surface which it touches at one point, in what direction does the surface react against the body?

4. Find the ratio of the effort to the weight on an inclined plane when the effort acts parallel to the plane.

5. If a power of 8 lb. applied parallel to an inclined plane support a weight of 17 lb., what is the thrust on the plane?

6. A weight rests on a smooth inclined plane. Show that the smallest force which will keep it in equilibrium must act along the plane.

7. Forces of 3, 5, and 7 lb., respectively, act on a particle at the centre of a circle, towards points on the circumference which divide it into three equal parts. Find the magnitude and direction of the force that will balance them.

8. A heavy uniform beam **AB** is supported at a point **C** by the prop **CD**, its extremity **A** pressing against a smooth wall **EF**. Determine the conditions of equilibrium.

9. If a, b, c be the breadths of the three faces of a wedge, supposed forced into a fissure in the usual manner, given the action P against its back a, determine the two reactions Q and R against its two sides b and c.

10. A uniform ladder rests against a smooth house wall at a slope to the ground of 45°. Draw a figure showing the directions of the forces acting on the ladder, and prove that the force exerted by the ground is $\sqrt{5}$ times the force exerted by the wall.

PART II.

MOMENTS AND PARALLEL FORCES.

CHAPTER V.

MOMENTS OF FORCES IN ONE PLANE.

60. **Forces tending to produce rotation.**—If a body is attached to a fixed axis or hinge about which it can turn freely, we can set the body in motion by applying to it a force in any direction not passing through the axis. And it is easy to verify by a few simple experiments that, the further off from the axis the force is applied, the more effect it has in turning the body.

Consider, for example, a door which can turn about its hinges. To open or shut the door, we apply a force to its handle in a direction perpendicular to the plane of the door, and at a considerable distance from the hinge. If we press against the woodwork of the door at a point very near the hinge, we shall have to exert a much greater effort to set the door turning, while if we lean against the edge of the door and push it directly against the hinge, it will not turn at all.

When a wheel is turned by a handle, we apply to it a force perpendicular to the arm of the handle. It is easy to verify that, the further the handle is from the centre of the wheel, the less will be the force required to turn the wheel, although the handle will have to be moved through a greater distance in each revolution.

In this chapter we shall consider the equilibrium of bodies under forces which tend to turn them about a fixed point or axis ; but shall not consider the actual motion of such bodies when the forces cause them to rotate.

61. DEFINITION.—*The* **moment of a force about a fixed point** *is the product of the measure of the force into the perpendicular distance of the point from its line of action.**

Thus, if P be the force, and if **OM** is drawn from any point **O** perpendicular on the line of action of P, the product $P \times \mathbf{OM}$ is called the **moment** of the force P about **O**.

The length **OM** may be called the **arm** of the moment. Therefore

moment = force × arm.

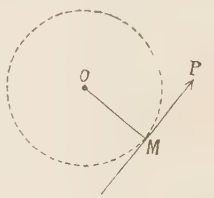


Fig. 58.

The product $P \times \mathbf{OM}$ becomes zero if either of its factors diminish to zero, that is,

if $P = 0$, or if $\mathbf{OM} = 0$.

In the latter case **O** is on the line of action of P.

Hence the moment of a force about a point vanishes when either—

(i.) the force itself is zero ;

(ii.) the line of action of the force passes through the point.

If the body be fixed so that it can only turn about **O**, the force P cannot set the body in motion if its line of action passes through **O**. (For, by the Principle of Transmission of Force, P may be supposed to act *at* **O**, and **O** is prevented from moving by the hinge or support.)

Hence, *when the moment of a force about a fixed point vanishes, the force has no tendency to turn the body about that point.*

On the other hand, if the moment is not zero, the force P cannot pass through **O**. But the reaction of the hinge or support passes through **O**. Hence these two forces cannot act in the same straight line, and the body cannot remain in equilibrium. It must therefore turn about **O**, since no other motion is possible.

* Notice that the words *moment* and *momentum* have entirely different meanings. *They must be carefully distinguished, for no connection whatever exists between them.*

From these and other considerations to be proved shortly, it may be inferred that the moment of a force about any point is a proper measure of the tendency of the force to produce rotation about that point.

62. Positive and negative moments.—In dealing with forces in one plane, it is convenient to regard the moment of a force about a point as positive or negative according to which way the force tends to rotate a body about that point.

Moments which tend to produce rotation in the *opposite* direction to that in which the hands of a watch go are considered **positive.**

Moments which tend to produce rotation in the *same* direction as the hands of a watch go are therefore to be regarded as **negative,** and a minus sign is prefixed to their amount.

Fig. 59.

Fig. 60.

Thus, in Figs. 58, 59, the moments about O of the forces indicated by the arrows are all positive ; but in Fig. 60 they are all negative.

The following rule is also convenient—

The moment of a force is
positive about all points on the **left** of its line of action,
negative ,, ,, ,, **right** ,, ,,
as seen by a person looking in the direction of the force.

In forming the **algebraic sum of the moments** of any number of forces, each moment is taken with its proper algebraic sign.

Thus, if two forces have equal moments about a point, but tend to produce rotation in opposite directions, their algebraic sum is zero, for one moment is a *plus* and the other a *minus* quantity. In particular, two equal and parallel forces tending in the same direction will have equal and opposite moments about a point midway between them, and the algebraic sum of the moments will be zero.

63. Geometrical representation of the moment of a force.

If a force be completely represented by a straight line, its moment about any point shall be measured by twice the area of the triangle which the straight line subtends at that point.

Let **AB** represent any force P, then shall the moment of P about **O** be represented by twice the area **OAB.**

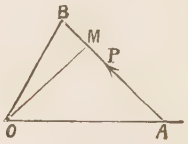


Fig. 61.

Draw **OM** perpendicular to **AB.** Then

area of $\triangle$**OAB** $= \frac{1}{2}$ base $\times$ altitude $= \frac{1}{2}$**AB** $\times$ **OM.**

Now **AB** represents the force P, therefore **AB** must contain P units of length. Hence

moment of P about **O** $= P \times$ **OM** $=$ **AB** $\times$ **OM** $= 2 \triangle$ **OAB.**

64. **If a given force is compounded with any force which passes through O, the moment about O will be unaltered.**

For if **AB** represents the given force P, and **AD** represents any force Q passing through **O,** their resultant R will be represented by **AC** the diagonal of the parallelogram **ABCD.** Since **OAD** and **BC** are parallel,

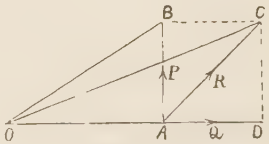


Fig. 62.

$\therefore$ $\triangle$**OAB** $=$ $\triangle$**OAC** ;

$\therefore$ moment of P about **O** $=$ moment of R about **O** ;

as was to be proved.

65. If a force P is applied at any point A, its moment about any point O is equal to the product

OA × resolved part of P perpendicular to OA.

For let **AC** represent the force P. Complete the rectangle **ABCD**, whose side **AD** passes through **O**. Then, by the Parallelogram of Forces, **AD, AB** represent the components of P along and perpendicular to **OA**.

Now $\triangle OAB = \triangle OAC$ (since they are on the same base and between the same parallels).

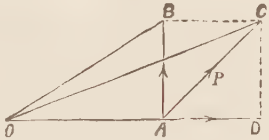


Fig. 63.

Therefore moment of P about **O**

$= 2\triangle OAC = 2\triangle OAB$

$= OA \times AB$

$=$ **OA** $\times$ resolved part of P perpendicular to **OA**.

66. **Another expression for the moment of a force.**—In the right-angled triangle **OAM** (Fig. 61) we have

$$\sin MAO = \frac{MO}{AO} = \frac{OM}{OA}; \quad \therefore \quad OM = OA \sin MAO;$$

$\therefore$ moment of P about $O = P \,.\, OM = P \,.\, OA \,.\, \sin MAO$.

Hence the moment of a force about a point is the product of the force, the distance of its point of application from the point, and the sine of the angle which this distance makes with the line of action of the force.

Cor. Since $P \sin MAO =$ resolved part of P perpendicular to **OA**, we have an independent proof of the property proved in § 65.

67. **The moments of two intersecting forces about any point in the line of action of their resultant are equal and opposite.**

Let two forces P, Q act in the lines **AB, AD**, and let **C** be any point in the line of action of their resultant R.

Complete the parallelogram **ABCD**.

Choose the scale of representation such that **AD** *represents the force Q in magnitude.**

Then, by the Parallelogram of Forces, **AB** and **AC** represent the forces *P*, *R*.

Since **ABCD** is a parallelogram,

$\therefore \quad \triangle \mathbf{CAB} = \triangle \mathbf{CDA}.$

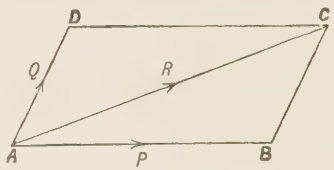


Fig. 64.

But the triangles **CAB**, **CAD** represent half the moments of *P* and *Q* about **C**, and these moments tend to turn about **C** in opposite directions.

Therefore the moments of *P*, *Q* about **C** are equal and opposite.

68. Equilibrium about a fixed point.—If a rigid body, moveable about a fixed point, is kept in equilibrium by two forces in any plane containing that point, the moments of these forces about the point will be equal and opposite.

For, if the forces balance, their resultant must pass through the fixed point. Hence the moments of the forces about that point must be equal and opposite, from § 67.

Either of the forces, if it were to act alone, would set the body turning round about the fixed point. Hence, since the body does not turn when both act, we are led to infer that the tendencies of the forces to produce rotation are equal and opposite.

Hence equal moments about a point represent equal tendencies to produce rotation about the point. Moreover, if a force be doubled, its moment about any point is also doubled; and it is natural to suppose that its tendency to produce rotation about any point is doubled. Hence we infer that the moment of a force about any point is a measure of its tendency to produce rotation about that point.

If two or more forces acting on any body have a single resultant, we should naturally expect the resultant to have the same tendency to produce rotation about any point as have the several forces acting simultaneously. We shall now prove that such is the case.

* This step of the proof should be carefully noted, as it is most important.

69. The algebraic sum of the moments of two intersecting forces about any point in their plane is equal to the moment of their resultant about that point (Varignon's Theorem).

Let two forces P, Q act along the lines **AB, AD.**

Let **O** be any point in the plane of the forces about which moments are to be taken.

Draw **OD** parallel to **AB** to meet **AD** in **D.**

Choose the scale of representation so that **AD** *represents* Q *in magnitude* (§ 10; also compare § 67), *and on the same scale let* **AB** *be the length which represents* P.

Let the parallelogram **ABCD** be completed, and join **AC**, so that **AC** represents the resultant R of P, Q.

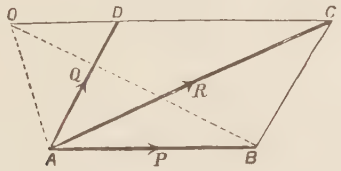


Fig. 65.

Case i.—If **O** lies *without* the angle **BAD**, as in Fig. 65, then the moments of P, Q, R about **O** are positive and measured by twice the triangles **OAB, OAD, OAC** respectively; hence we have to show that

$$2\triangle \mathbf{OAB} + 2\triangle \mathbf{OAD} = 2\triangle \mathbf{OAC}$$

Now $$\triangle \mathbf{OAB} = \triangle \mathbf{DAC}$$

(for each is equal to the triangle **ABC**);

$$\therefore\ 2\triangle \mathbf{OAB} + 2\triangle \mathbf{OAD} = 2\triangle \mathbf{DAC} + 2\triangle \mathbf{OAD} = 2\triangle \mathbf{OAC},$$

or moment of P + moment of Q = moment of force R.

Case ii.—If **O** lies *within* the angle **BAD**, say between **AC** and **AD**, as in Fig. 66, the moment of Q is negative

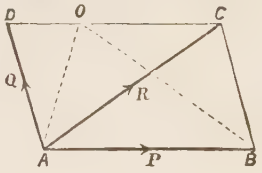


Fig. 66

and is represented by *minus* twice the area **ODA**, and we have to show that

$$2\triangle\mathbf{OAB} - 2\triangle\mathbf{ODA} = 2\triangle\mathbf{OAC}.$$

Now, as before, $\triangle\mathbf{OAB} = \triangle\mathbf{DAC}$,

$$\therefore 2\triangle\mathbf{OAB} - 2\triangle\mathbf{ODA} = 2\triangle\mathbf{DAC} - 2\triangle\mathbf{ODA} = 2\triangle\mathbf{OAC}.$$

Therefore, making allowance for the positive and negative signs of the moments themselves, we have algebraically

moment of P + moment of Q = moment of force R,

as before.

Alternative Proof.—By § 65 (using Figs. 26, 27, p. 29), Algebraic sum of moments of P, Q about **O**

= **OA** × algebraic sum of resolved parts of P, Q perpendicular to **OA**

= **OA** × resolved part of R perpendicular to **OA** (§ 30)

= moment of R about **O**.

Observations.—The student should write out full proofs of the theorem, with figures, suited to the following cases :—(*a*) When **O** lies within the angle **BAC**. (*b*) When **O** lies within the vertically opposite angle to **BAD** formed by producing **BA, DA**.

[The more advanced student will find that the proof of Case i. may be made to include every case by making the following convention as to the sign of the area of a triangle, *viz.*, that the area is to be regarded as *positive* or *negative* according to whether in going round the sides in the order of the letters used in naming the triangle we always have the triangle on our *left* or *right*. With this convention, $\triangle$ **ODA** will represent an area equal and opposite to $\triangle$ **OAD**, so that $\triangle$ **ODA** + **OAD** = 0, and the proofs of the two cases given above will be found to be identical.]

70. **Generalisation.**—*If any number of forces act at a point in one plane, the algebraic sum of their moments about any point in that plane is equal to the moment of their resultant.*

This may be proved by combining two of the forces and then combining their resultant with a third force, and so on.

Cor. 1. If any number of forces acting at a point in one plane are in equilibrium, the algebraic sum of their moments about any point in the plane is zero.

For their resultant is zero; therefore its moment about any point is zero, and therefore, by above, the sum of all the moments is zero.

Cor. 2. If the algebraic sum of the moments of any number of forces about a point O in their plane vanishes, then *either* R must be zero, or the *arm* of R must be zero. That is *either* the forces are in equilibrium *or* they have a resultant passing through O.

71. **Conditions of Equilibrium of Coplanar Forces.**—Although it is impossible here to fully investigate the conditions of equilibrium of forces acting *on a rigid body* (that is to say, not all at one point) *in one plane*, yet it is well to note the main results.

To the two conditions of equilibrium given in § 35, must be added a third like Cor. 1 of § 70.

Hence the conditions of equilibrium, when any number of forces in one plane act upon a rigid body, are as follows :—

(*a*) The sums of the resolved parts of the forces along two directions in the plane must be severally zero.

(*b*) The algebraic sum of the moments of the forces about any point in their plane must be zero.

[The former conditions prevent any motion of *translation*, the latter prevents any motion of *rotation*.

The conditions for parallel forces (see Chap. VI.) will be found to be included in these.]

72. **Difference between the moment and the work of a force.**—If we compare the definitions of § 37 and § 61, we shall observe that both the moment of a force about a point and the work of a force are products of forces into lengths of straight lines. But the moment of a force about a point is the product of a force into a straight line **at right angles to it,** while work is the product of a force into a straight line **in the same direction.**

Thus, let a force P be applied to a particle at **A**. Then

moment of P about **O**

$= P \times$ **OM** (Fig. 67)

$=$ **OA** $\times$ resolved part of P *perpendicular* to **OA**;

work done in moving the particle from **A** to **O**

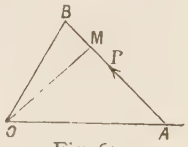


Fig. 67.

$= P \times$ **AM** $=$ **AO** $\times$ resolved part of P *along* **AO**.

It is also to be observed that the moment of a force is a purely statical idea, but *work is only done when motion takes place.*

73. *Examples.*—(1) Fig. 68 represents a uniform rod **AB**, of weight 10 lb. and of length 4 ft., hinged at **A** to a wall and carrying a weight of 12 lb. attached to it at a distance of 1 ft. from **B**. It is supported in a horizontal position by means of a string **BC**, which is fastened to a point **C** in the wall vertically above **A**, and which forms an angle of 30° with the rod **AB**. Find the tension of this string.

Consider the equilibrium of the rod. The forces acting upon it are :—

The weight 10 lb. acting vertically down at the middle point **G**,

A downward pull of 12 lb. at **F**,

R, the reaction of the hinge at **A** (of unknown magnitude and direction),

T, the tension of the string, acting along **BC**.

Of these forces :—

R has no moment about **A**, since its line of action passes through **A** (cf. § 61).

The forces 10 and 12 lb. tend to rotate the rod round **A** in a clockwise direction; and the force T prevents this rotation.

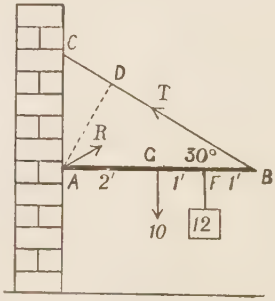


Fig. 68.

Hence the moment (or turning power) of the force T about **A** must balance the moments of the forces 10 and 12 lb. about **A**.

i.e. $$T \times \mathbf{AD} = 10 \times \mathbf{AG} + 12 \times \mathbf{AF}$$

(where **AD** is perpendicular to **BC**, cf. Def. § 61).

Now $$\mathbf{AD} = \mathbf{AB} \cdot \sin \mathbf{ABD} = 4 \cdot \sin 30° = 2 \text{ feet.}$$

Thus we have $$T \times 2 = 10 \times 2 + 12 \times 3,$$

whence $$T = 28 \text{ lb.}$$

Note that in this example we have not used the algebraic signs of the moments, but have regarded them all as positive. We then equate the sum of the moments that tend to turn in one direction to the sum of those that tend to turn in the opposite direction.

(2) In Fig. 68*a*, **ABCD** is a rectangular block of stone weighing 600 lb., standing on a rough horizontal plane, **AB** = 5 feet, **AD** = 12 feet. What horizontal force must be applied at **D** to tilt the block round **A**; assuming that the ground is rough enough to prevent slipping?

[Note that strictly **ABCD** is not the front face of the block, but the middle section of the block, as represented in the larger diagram.]

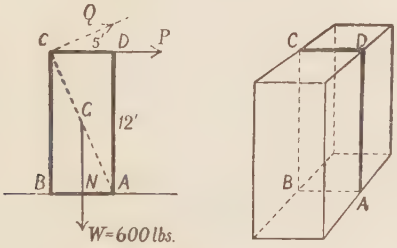


Fig. 68*a*.

If *G* bisects **AC** and **N** bisects **AB**, then **GN** is the line of action of *W*.

Now the force *P* is trying to make the block turn round the point **A** (or more correctly round the edge of which **A** is the middle point). The force *W* will, if possible, prevent the block from turning.

Thus the block will be *on the point of tilting*, if the moment of *P* about **A** is equal to the moment of *W* about **A**.

Thus we have $P \times \mathbf{AD} = W \times \mathbf{AN}$,

i.e. $P \times 12 = 600 \times 2\frac{1}{2}$,

whence $P = 125$ lb.

Thus any force greater than 125 lb. will tilt the block.

(3) In the last question, what is the smallest force which could tilt the block round **A**? At what point and in what direction must it be applied?

Suppose a force *Q* applied at some point of the block; and let the length of the perpendicular drawn from **A** to the line of action of *Q* be *q*. Then the moment of *Q* about **A** is $Q \times q$.

As before, the block is on the point of tilting when the moment of Q about **A** is equal to the moment of W about **A**,

i.e. when $Q \times q = 600 \times 2\frac{1}{2} = 1500$

or $Q = 1500/q$.

Thus in order to make Q as small as possible, we must make q as large as possible. This requires

(i) that Q should be applied at a point as far as possible from **A**, viz. at **C**,

(ii) that the line of action of Q should be perpendicular to **AC**.

For in this case $q =$ **AC**, and it is obviously impossible to make q greater than **AC**, since the point of application must be somewhere on the block.

We now have $Q = 1500/\mathbf{AC}$,

whence (by Euc. I. 47) $Q = 1500/13$ lb.

(4) Forces P, Q, R act along the sides **BC, CA, AB** of a triangle **ABC**. Find, by Varignon's Theorem, the line of action of the resultant. (Fig. 68*b*).

As usual in Trigonometry, let a, b, c be the sides of the triangle, and **A, B, C** its angles. Let **HK** be the required line of action of the resultant Z; let $\mathbf{BH} = x$, $\mathbf{CK} = y$. In order to determine the position of the line **HK** it will be sufficient to find x and y.

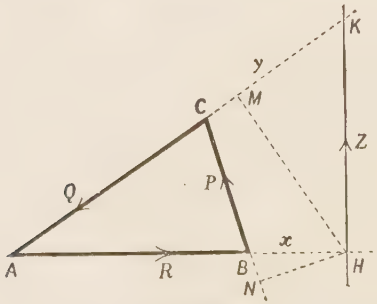


Fig. 68*b*.

Draw **HM** and **HN** perpendicular to **CA** and **BC** respectively.

Now the algebraic sum of the moments of P, Q, R about **H**

$=$ moment of Z about **H** (§ 70.)

$= O$ (§ 61.)

Thus $-P \times \mathbf{HN} + Q \times \mathbf{HM} + R \times O = O$ ..... (i)

Again $\mathbf{HN} = \mathbf{HB} \cdot \sin \mathbf{HBN} = x \sin \mathbf{B}$,

and $\mathbf{HM} = \mathbf{HA} \cdot \sin \mathbf{HAM} = (c + x) \sin \mathbf{A}$.

Thus from (i) $$- P \cdot x \sin \mathbf{B} + Q (c + x) \sin \mathbf{A} = 0,$$

whence $$x = \frac{Q c \sin \mathbf{A}}{P \sin \mathbf{B} - Q \sin \mathbf{A}}.$$

Similarly we can show that

$$y = \frac{R b \sin \mathbf{A}}{P \sin \mathbf{C} - R \sin \mathbf{A}}.$$

Summary of Results.

Moment of force P about point **O**

$= P \times$ (perpendicular distance of P from **O**). (§ 61.)

Moments are *positive* which tend in *opposite* direction to hands of clock. (§ 62.)

Moments are positive about points on *left-hand* of force.

If force P is represented by **AB**, then its moment about **O**

$= 2 \cdot \triangle \mathbf{OAB}$. (§ 63.)

$= \mathbf{OA} \times$ (resolved part of P *perpendicular* to **OA**). (§ 65.)

Varignon's Theorem or the Equation of Moments.—The moment of resultant of two (or more) forces = algebraic sum of moments of its components. (§ 69.)

[Proved for parallel forces in the next chapter, § 80.]

For equilibrium about a fixed point, the algebraic sum of the moments must be zero. (§ 68.)

EXAMPLES V.

(*Selected Problems* :—1, 2, 5, 6, 9, 11, 12, 14, 15, 16, 17, 18, 19, 22, 23, 27, 28.)

1. Find the moment round a point **O** of a force of 3 lb. acting at a point **A** along a line **AB**, where **OA** is equal to 10 in., and the angle **OAB** equals (i.) 90°, (ii.) 45°, (iii.) 135°, (iv.) 120°.

2. **ABC** is a triangle, right-angled at **C**, and having the angle **B** equal to 60°. Forces 3, 4, 5 act along **AB**, **BC**, **CA**, respectively. Find the moment of each round the opposite angular point.

3. **OA**, **OB** are chords, 4 and 5 in. in length, of a circular disc **OACB**, whose diameter **OC** is 6 in. If forces of 3 and 4 lb. act from **O** along these chords respectively, find how the disc will begin to move, the point **C** being fixed.

4. Explain why a boy, pulling hold of the rim of a garden roller at its top point, can pull back a much stronger boy catching hold of the handle. If the first boy pulls the roller back in a horizontal direction, and the second pulls the handle in a direction making an angle of 30° with the horizon, compare the forces which they exert on it if no motion takes place.

5. Forces act along all the sides but one of a plane polygon, and are represented by the sides in magnitude ; prove that, when they all act the same way round, their resultant is a force parallel to the remaining side and represented by it in magnitude. Prove, also, that the line of this resultant forms with the remaining side a parallelogram whose area is twice the area of the polygon.

6. A horizontal rod 8 ft. long has a weight of 1 lb. at one end, upward forces of 2 and 3 lb. act at distances 2 and 6 ft. from that end, and a weight of 4 lb. hangs from the other end. Taking the first end as the left-hand end in a figure, write down the moments of the forces about each end of the rod and about its middle point, prefixing the proper sign to each. Also find the algebraic sums of the moments about these points.

7. Forces of 1, 3, 5, 7, 9, 11 lb. act along the sides **AB, BC,** etc., of a regular hexagon **ABCDEF**. Find the moment of each force about the point **A.**

8. Draw an equilateral triangle **ABC,** and suppose each side to be 4 ft. long ; a force of 8 units acts from **A** to **B,** and a force of 10 units from **C** to **A.** (*a*) Find the moment of each force with reference to the middle point of **BC.** (*b*) Find a point with reference to which the forces have equal moments of opposite signs.

9. Draw a square **ABCD** ; suppose forces of P and Q units to act from **A** to **B** and from **A** to **D,** respectively. Find the perpendicular distance of the line of action of their resultant from **C.**

10. Two forces of 5 lb. and 12 lb., respectively, act at right angles. Find the locus of the points in their plane round which the sum of their moments is $3\frac{1}{2}$. (*Apply Varignon's Theorem.*)

11. A system of forces in a plane is such that the sum of their moments about a point **A** in that plane vanishes. If the forces are not in equilibrium, what do we know about their resultant ?

12. A square whose side is 3 ft. long is divided into 9 smaller squares, each 1 ft. square. Forces of 3, 4, 5, 6 lb. act along the sides

of the middle square taken in order, in such directions that their moments about any point inside that square are all positive. Write down the moments of each force about each of the four angular points of the large square, prefixing the proper sign to each, and find the algebraic sum of the moments about each of these four points.

13. ABCD is a square. Equal forces (P) act from D to A, A to B, and B to C, respectively, and a fourth force $2P$ acts from C to D. Find a point such that, if the moments of the forces are taken with respect to it, the algebraical sum is zero.

14. A lamina is free to turn in its own plane about a point O; at a point A two forces P and Q are acting on the body in its own plane. If P and Q tend to rotate the body in opposite directions, and if the perpendicular distances of the lines of action from O are p and q respectively, prove *without using the theory of moments* that the condition for equilibrium is

$$P \times p = Q \times q.$$

15. A uniform rod weighing 10 lb. is hinged at its lower end, and is supported at an angle of 60° with the vertical by a horizontal string attached to its upper end. If the upper end also carries a weight of 40 lb., find the tension of the string.

16. A rod AB of weight W is hinged at A to a vertical wall. A weight V is suspended from B, and a string BC connects the rod with a point C vertically over A. Find the tension of this string, given AC = BC.

17. A cube of weight 100 lb. rests on a horizontal plane which is rough enough to prevent slipping. Find the smallest *horizontal* force which will make the cube tilt if applied perpendicular to the middle of one of the upper edges. Also find the smallest force in any direction which will tilt the cube.

18. A uniform board of weight W in the shape of a regular pentagon is standing on a rough horizontal plane. Show that the smallest *horizontal* force which will cause the board to tilt if applied at the vertex in the plane of the board is $W \tan 18°$; and that the smallest force in the plane of the board which will tilt it is $W \sin 18°$.

19. A string wound round the rim of a wheel, which is free to turn about its centre **O** in a vertical plane, passes over a pulley and supports a weight Q. A weight P is attached to a point **A** on the rim of the wheel. Prove that in the position of equilibrium **OA** makes with the vertical an angle $\sin^{-1} Q/P$.

20. **AOB** is a weightless rod bent to form an angle α at **O**. Weights P and Q are attached at **A** and **B**, and the whole is suspended from **O**. Given $\mathbf{OA} = a$, $\mathbf{OB} = b$, prove that if θ be the inclination of **OA** to the horizontal,

$$\tan\theta = \frac{Pa}{Qb\sin\alpha} + \cot\alpha.$$

21. A uniform rod of weight W and length $2a$ is resting with its middle point supported by the highest point of a perfectly rough sphere of radius r. Weights P and Q are attached to its extremities, and it is left to find its equilibrium position. Prove that in this position the circular measure of the inclination of the rod to the horizontal is

$$\frac{a(Q-P)}{r(P+W+Q)}.$$

22. Forces P, Q, R act along the sides **BC**, **CA**, **AB** of a triangle. The line of action of their resultant divides **BC** in the ratio $k : l$, and **AC** in the ratio $h : l$. Prove by § 70 that

$$P : Q : R = ha : kb : -lc,$$

where a, b, c are the sides of the triangle.

23. Prove by § 70, that if P, Q, R are forces acting along the sides **BC**, **CA**, **AB** of a triangle, the condition that the line of action of their resultant passes through the intersection of the medians is

$$Pbc + Qca + Rab = 0.$$

24. If in the preceding problem the line of action of the resultant passes through the centres of the inscribed and circumscribed circles of the triangle, show that

$$P : Q : R = \cos B - \cos C : \cos C - \cos A : \cos A - \cos B.$$

25. A particle **P** is attracted to two fixed points **A** and **B** by forces represented by $\mu\mathbf{PA}$, $\mu'\mathbf{PB}$. Find the locus of **P** such that the sum of the moments of the forces about a given point **O** may be equal to a given constant.

26. Assuming that the moment of the resultant of two forces is equal to the algebraical sum of the moments of the two forces about

the same point, prove the parallelogram of forces for the magnitude and direction of the resultant.

27. A uniform rod of weight W and length l is hinged at one end. To the other end is fastened a string which passes over a pulley, which is vertically over the hinge, at a height l. Prove that in the position of equilibrium the inclination of the rod to the vertical is $2 \sin^{-1} P/W$, where P is the tension of the string.

28. A cube of wood, of edge 6 inches, weighs 9 lb. A metal rod, each inch of which weighs ·1 lb., is to be fastened to the top face of the cube, so that one end is at the middle of one edge; while the rod projects beyond, and bisects, the opposite edge. Find the greatest length of the rod which will not make the cube tilt over.

CHAPTER VI.

PARALLEL FORCES.

74. **"Like" and "unlike" parallel forces.**—In the preceding chapters we have considered the equilibrium of forces whose directions intersect one another. In the practical applications of mechanics, however, parallel forces are of even more frequent occurrence than intersecting forces.

The following definitions will be required :—

DEFINITION.—Parallel forces which tend in the same direction are said to be **like.** Those which tend in opposite directions are said to be **unlike.**

If a force acting in one direction be regarded as *positive*, it is convenient to regard any unlike force as *negative.*

Thus, if there be forces of 28 lb. acting upwards and 56 lb. acting downwards, the two forces will be unlike, and if we consider the upward direction as positive, the downward one will be negative, and the complete expressions for them will be + 28 lb. and − 56 lb., respectively.

75. **Composition of parallel forces (preliminary observations).**

When two parallel forces P, Q act at any two points **A, B** of a rigid body, their resultant, if it exists, may be deduced from the resultant of two intersecting forces by making use of the theorem of § 51. If, at **A** and **B**, we apply to the body equal and opposite forces F and $-F$, both acting in

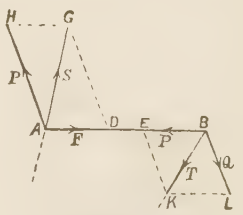


Fig. 69.

the straight line **AB**, these will not affect the equilibrium of the body as a whole. Now compound P with F, and Q with $-F$.

The two former are equivalent to a single force S at **A**, whose direction is *within* the angle **BAP**, and the two latter to a single force T at **B** *within* the angle **ABQ**.

Case i. If P, Q be *like* forces, the angles they form with **AB** both lie on the same side of **AB** (Fig. 70); hence the forces S, T, on being produced, intersect one another in a point **M** *between* the forces P, Q. Therefore S and T must be equivalent to a single resultant force passing through **M**. This will also be the resultant of P, Q, and will be completely determined in § 77.

Case ii. If P, Q be *unlike and unequal* forces, suppose P to be the greater, and construct the Parallelograms of Forces **ADGH**, **BEKL**, so that **AG**, **BK** represent S, T respectively (Fig. 69).

Since $P > Q$, **DG** > **EK**. Hence, by placing the triangle **DAG** with its vertices **A**, **D** on **B**, **E** (as in proof of Euc. I. 4), we see that $\angle$ **DAG** > $\angle$ **EBK**, and it readily follows from Euclid's twelfth axiom that **BK** produced will meet **GA** produced through **A** in a point **M** on the side of the line of action of P remote from Q (Fig. 71). Therefore S and T or P and Q have a single resultant through **M**, which will be completely determined in § 78.

Case iii. If P, Q are *unlike* and *equal* forces the method fails, for the forces S, T are themselves parallel, as is evident from the construction of Case ii. Two equal unlike parallel forces are said to constitute a **couple.***

76. In the present chapter we shall deal with parallel forces that do not form couples. We shall, therefore, always assume that they have a resultant or else that three or more of them are in equilibrium.

* Hence a *couple* in Statics does *not* simply mean "two of anything" as in ordinary language.

77. To find the resultant of two *like* parallel forces.

Let P, Q be the two parallel forces, **A**, **B** any two points on their lines of action which we may suppose to be their points of application.

At **A** introduce a force of magnitude F along **AB**, and at **B** introduce an equal and opposite force F along **BA** (that is, $-F$ along **AB**); these two forces will balance each

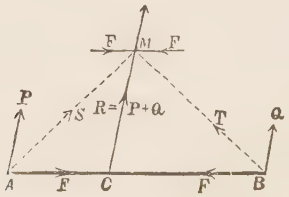


Fig. 70.

other, and will not affect the resultant. Let S be the force compounded of P and F, T the force compounded of Q and $-F$.

Then the forces S and T will intersect at a point **M** between the forces P, Q, and will have a resultant R acting at **M**, which will also be the resultant of P and Q.

Draw **MC** parallel to P or Q, cutting **AB** in **C**.

Replace the force S at **M** by its components along **CM** and parallel to **AC**. These components are, of course, the same as when S acted at **A**, and are therefore P, F. Similarly, replace the force T at **M** by its components along **CM** and parallel to **BC**; these components are Q, F.

Thus the two forces P, Q or S, T are together equivalent to forces $P + Q$ acting along **CM**, and $F - F$ or zero parallel to **AC**.

Hence the resultant of P, Q is P + Q *acting along* CM ; or, in words—

The *magnitude* of the resultant is the sum of the components.

The *direction* of the resultant is parallel to the components, and in the same sense.

The *position* of the resultant lies between the two forces, and may be found as follows :—

Since the forces at **A** are parallel to the sides of $\triangle$**ACM**, therefore, by the Triangle of Forces,

$$\frac{P}{F} = \frac{\mathbf{CM}}{\mathbf{AC}}, \quad \text{or} \quad P \times \mathbf{AC} = F \times \mathbf{CM}.$$

Similarly,* since the forces at **B** are parallel to the sides of $\triangle$**BCM**,

$$\frac{Q}{F} = \frac{\mathbf{CM}}{\mathbf{CB}}, \quad \text{or} \quad Q \times \mathbf{CB} = F \times \mathbf{CM}.$$

$$\therefore \quad P \times \mathbf{AC} = Q \times \mathbf{CB}.$$

If **AB** is taken at right angles to the forces, this relation expresses the fact that **the moments of P, Q about C are equal and opposite.**

We may also write the above relation,

$$\frac{\mathbf{AC}}{\mathbf{CB}} = \frac{Q}{P},$$

which shows that **AC** is greater or less than **CB**, according as Q is greater or less than P. Hence

The resultant is nearer to the greater force.

It divides the line AB into parts which are in the inverse ratio of the forces.

---

* The sides of $\triangle$ **ACM** do not represent the forces at **A** *on the same scale* that the sides of $\triangle$ **BCM** represent the forces at **B**. In the former triangle, **AC** represents F and **CM** represents P. In the latter, **BC** represents F and **CM** represents Q. Hence *the scales of representation in the two triangles are different.*

Note also that, *algebraically*, we should strictly have $Q\ (-F) =$ **CM**/**BC**, and, since **CB** $= -$ **BC**, this gives $Q/F =$ **CM**/**CB**, as stated.

78. To find the resultant of two unequal *unlike* parallel forces.

Let P, Q be the forces, P being the greater, **A, B** any two points on their lines of action. Introduce two equal and opposite forces F, $-F$ at **A, B** acting along **AB.** Let S be the force compounded of P and F, T the force compounded of Q and $-F$. Then the forces S, T lie in the alternate angles at **A** and **B,** and their lines of action will intersect at a point **M** on the side of P remote from Q (§ 76). Draw **MC** parallel to P, cutting **AB** in **C.**

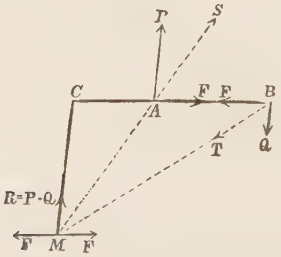


Fig. 71.

Replace the force S at **M** by its components P, F along **MC** and parallel to **AB,** and replace the force T at **M** by its components Q, F along **CM** and parallel to **BA.** Thus the two forces P, Q or S, T are together equivalent to $P - Q$ along **MC,** and $F - F$ or zero parallel to **AB.**

Hence the resultant of P, Q is $P - Q$ acting along **MC** ;

or, in words—

The *magnitude* of the resultant is the difference of the components.*

* Since Q is unlike to P, its complete algebraic expression is strictly $-Q$, and the resultant $P + (-Q)$ is still the *algebraic sum* of the components, although it is the difference of their numerical values.

The *direction* of the resultant is parallel to, and in the sense of, the greater component.

The *position* of the resultant lies on the side of the greater force which is remote from the lesser force, and is found thus :—

Since the forces at **A** are parallel to the sides of $\triangle$**MCA**,

$$\frac{P}{F} = \frac{\mathbf{MC}}{\mathbf{CA}}, \quad \text{or} \quad P \times \mathbf{CA} = F \times \mathbf{MC}.$$

Similarly, since the forces at **B** are parallel to the sides of $\triangle$**BCM**,

$$\frac{Q}{F} = \frac{\mathbf{MC}^*}{\mathbf{CB}}, \quad \text{or} \quad Q \times \mathbf{CB} = F \times \mathbf{MC}.$$

$$\therefore \quad P \times \mathbf{CA} = Q \times \mathbf{CB};$$

or the moments of P, Q about **C** are equal and opposite.

We may also write this relation

$$\frac{\mathbf{CA}}{\mathbf{CB}} = \frac{Q}{P};$$

and, since we have taken Q less than P, **CA** is less than **CB**, which affords another proof that **C** lies on **BA** produced beyond **A**. Hence

The resultant divides the line AB externally into parts which are in the inverse ratio of the forces.

79. The resultant of *any number* of parallel forces is parallel to the forces and equal to their algebraic sum.

Let P, Q, R, S be the parallel forces taken with their proper signs. Then the two forces P, Q have a resultant $P + Q$ parallel to them ; the forces $P + Q$ and R have a resultant $P + Q + R$ parallel to them, which is therefore the resultant of the *three* forces P, Q, R ; the forces $P + Q + R$ and S have a resultant $P + Q + R + S$ parallel to them, which is therefore the resultant of the *four* forces P, Q, R, S, and so on.

[This is true whether the forces are positive or negative ; see footnote on opposite page.]

* Strictly $Q/(-F) = \mathbf{CM}/\mathbf{BC}$.

80. The algebraic sum of the moments of two parallel forces (not forming a couple) about any point in their plane is equal to the moment of their resultant about that point.

(Varignon's Theorem for Parallel Forces.)

It has been proved in § 69 that the sum of the moments of two *intersecting* forces about any point is equal to the moment of their resultant. Hence it only remains to extend the theorem to the case when the forces are parallel.

Let P, Q be the two parallel forces, R their resultant, O any point in their plane. Through O draw a straight line perpendicular to the forces P, Q, R and cutting them in **A**, **B**, **C**, respectively. In this straight line apply two equal and opposite forces F, $-F$. Let S, T be the forces

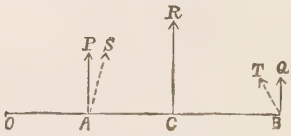


Fig. 72.

compounded of P, F and Q, $-F$, respectively; then R, the resultant of the parallel forces P, Q, is also the resultant of the forces S, T. Now the forces S, T intersect one another (since P, Q do not form a couple).

$\therefore$ moment of R about **O** = algebraic sum of moments of S, T.

Now moment of R about **O** $= R \times$ **OC**.

Also P, Q are the resolved parts of S, T perpendicular to **OB**.

$\therefore$ moment of S about **O** $= P \times$ **OA**;

moment of T about **O** $= Q \times$ **OB**.

$\therefore \quad R \times \mathbf{OC} = P \times \mathbf{OA} + Q \times \mathbf{OB}$;

or moment of R about **O** = algebraic sum of moments of P, Q about **O**.

This relation is sometimes called the **Equation of Moments.**

Alternative Proof.—With the notation of Fig. 72, algebraic sum of moments of P, Q about **O**

$$= P \times \mathbf{OA} + Q \times \mathbf{OB}$$
$$= P\,(\mathbf{OC} - \mathbf{AC}) + Q\,(\mathbf{OC} + \mathbf{CB})$$
$$= (P + Q)\,\mathbf{OC} + (Q\,.\,\mathbf{CB} - P\,.\,\mathbf{AC})$$
$$= (P + Q)\,\mathbf{OC} + 0\ (\S\ 77) = R \times \mathbf{OC}$$
$$=$$ moment of R about **O**.

This proof is perfectly general provided that the forces P, Q, R and the lengths **OA**, **OB**, **OC** are subject to the usual conventions regarding signs.

Cor. 1. When three parallel forces are in equilibrium, the algebraic sum of their moments about any point in their plane is zero.

For each force is equal and opposite to the resultant of the other two; therefore its moment is equal and opposite to the moment of that resultant, *i.e.* to the sum of the moments of the other two forces.

Cor. 2.—When any number of coplanar forces* act on a rigid body, the algebraic sum of their moments about any point in their plane is equal to the moment of their resultant.

Consider two of the forces. Whether these intersect or are parallel, algebraic sum of their moments about **O** = moment of their resultant.

Combine this resultant with a third force; then,
algebraic sum of moments of the three
= moment of third force + moment of resultant of first two
= moment of resultant of all three forces,
and so on, till all the forces have been compounded together.

Observation.—In finding the position of the resultant of two or more parallel forces, it is usually better to apply the present principle by "taking moments," *i.e.* writing down the "equation of moments" about any convenient point, instead of employing the results proved in § 77 or § 78. As a rule it is most convenient to take a point in the line of action of one of the forces, because the moment of that force is then zero. We may, however, take moments about a point on the resultant, or about any point whatever.

* *I.e.* forces in one plane.

Examples.—(1) To find the resultant of two like forces of 5 lb. and 4 lb. applied at the ends of a rod 3 ft. long perpendicular to its length.

Let **AB** be the rod, **C** the point at which the resultant cuts it. (Fig. 72 may be used, taking $P = 5$ lb. and $Q = 4$ lb.)

Since the forces are like, their resultant $= 4 + 5$ lb. $= 9$ lb.

Take moments about **A**. Then the moment of 9 lb. at C equals the sum of the moments of 5 lb. at **A** and 4 lb. at B, whence, taking a foot as the unit of length,

$$9 \times \mathbf{AC} = 5 \times 0 + 4 \times \mathbf{AB} = 0 + 4 \times 3 = 12;$$
$$\therefore \quad \mathbf{AC} = \tfrac{12}{9} \text{ ft.} = 1\tfrac{1}{3} \text{ ft.} = 16 \text{ in.}$$

Therefore the resultant acts at a distance of 16 in. from the 5 lb. force.

(2) To find the resultant of two *unlike* forces of 5 lb. and 4 lb. applied at points distant 3 ft. apart.

Let **A, B, C** be the points of application of the forces and their resultant.

Since the forces are unlike, their resultant $= 5 - 4$ lb. $= 1$ lb.

Taking moments about **A**, the equation of moments gives

$$1 \times \mathbf{AC} = 5 \times 0 - 4 \times \mathbf{AB} = 0 - 4 \times 3 = -12;$$
$$\therefore \quad \mathbf{AC} = -12 \text{ ft.} \quad \text{or} \quad \mathbf{CA} = 12 \text{ ft.}$$
$$\therefore \quad \mathbf{CB} = 12 + 3 = 15 \text{ ft.}$$

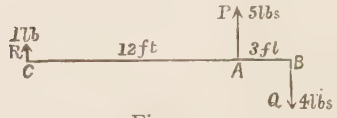


Fig. 73.

Therefore the resultant acts at distances of 12 and 15 ft. from the 5-lb. and 4-lb. forces respectively.

(3) To find the parallel forces which must be applied to a bar 9 ft. long in order that their resultant may be a force of 12 lb. acting at 2 ft. distance from one end.

Let P, Q be the required forces, **BC** the bar, **A** the point at which the resultant acts.

Taking moments about **C**, we have

$$P \times \mathbf{CB} + Q \times 0 = 12 \times \mathbf{CA};$$
$$\therefore \quad P \times 9 = 12 \times 2 = 24;$$
$$\therefore \quad P = \tfrac{24}{9} \text{ lb.} = \tfrac{8}{3} \text{ lb.} = 2\tfrac{2}{3} \text{ lb.}$$

Also $$P + Q = 12 \text{ lb.};$$
$$\therefore \quad Q = 12 - 2\tfrac{2}{3} = 9\tfrac{1}{3} \text{ lb.}$$

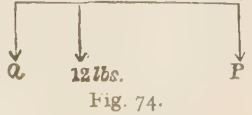


Fig. 74.

81. Conditions of equilibrium of three parallel forces.

If three parallel forces are in equilibrium, each force is proportional to the distance between the other two.

Let P, Q, R be the three forces, of which P, Q are *like* forces. Let a straight line be drawn, cutting them in **A, B, C.**

In the figure of § 77, reverse the force R so as to make it the equilibrant instead of the resultant of P and Q.

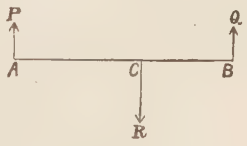


Fig. 75.

R has thus been changed in direction only, not in magnitude nor line of action. Hence we see that, when three parallel forces are in equilibrium, the middle one is equal to the sum of the two outer ones, and acts in the opposite sense. Also

$$\frac{P}{BC} = \frac{Q}{CA},$$

and, by a well-known theorem*, each member

$$= \frac{P+Q}{BC+CA} = \frac{P+Q}{BA}.$$

Now R is equal and opposite to $P + Q$.

Also **AB** is equal and opposite to **BA**.

Hence the conditions of equilibrium may be written in the symmetrical form,

$$\frac{P}{BC} = \frac{Q}{CA} = \frac{R}{AB} \quad \ldots\ldots\ldots\ldots \quad (1).$$

This result is analogous to the Triangle of Forces, and may be deduced from it.

[Notice that P stands over the length which does *not* contain the letter **A**; Q, R over the lengths which do not contain **B, C**, respectively.]

* If $\frac{a}{b} = \frac{c}{d}$, each fraction $= \frac{a+c}{b+d} = \frac{\text{sum of numerators}}{\text{sum of denominators}}$.

This relation holds good for the directions as well as the magnitudes of the forces.

Thus, taking the points in the order of Fig. 75, since **A, B, C** lie in a straight line, the two lengths **BC, CA** are together equal and opposite to the third length **AB**, so that, algebraically, $\mathbf{BC} + \mathbf{CA} + \mathbf{AB} = O$.

And the two forces P, Q are together equal and opposite to the third, so that, algebraically,

$$P + Q = -R, \quad \text{or} \quad P + Q + R = O.$$

The conditions of equilibrium of three parallel forces may also be stated thus :

The two extreme forces act in the same direction, and the middle force acts in the reverse direction and is equal and opposite to their sum. Also each force is proportional to the distance between the other two.

Example.—A rod 10 ft. long, whose weight W acts at its middle point, has a weight $3W$ attached to one end. To find at what point it must be supported in order to rest balanced.

Let **A** be the end, **C** the middle point of the rod, **D** the point of support, and let R denote the reaction at **D**.

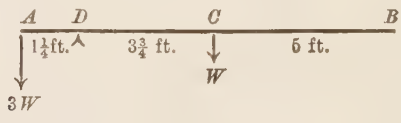


Fig. 76.

Then, by the conditions of equilibrium,

$$\frac{W}{\mathbf{AD}} = \frac{3W}{\mathbf{DC}} = \frac{R}{\mathbf{CA}};$$

$$\therefore \quad \mathbf{DC} = 3\mathbf{AD};$$

$$\therefore \quad \mathbf{AC} = \mathbf{AD} + \mathbf{DC} = 4\mathbf{AD} \quad \text{and} \quad \mathbf{AD} = \tfrac{1}{4}\mathbf{AC}.$$

But $\mathbf{AC} = 5$ ft. Therefore

$$\mathbf{AD} = \tfrac{5}{4} \text{ ft.} = 1\tfrac{1}{4} \text{ ft.}, \quad \mathbf{DC} = 3\mathbf{AD} = 3\tfrac{3}{4} \text{ ft.}$$

Hence the rod balances about a point $1\frac{1}{4}$ ft. from the end at which $3W$ is attached.

82. Experimental verification.—(*a*) DETAILS OF EXPERIMENT.—To verify, experimentally, the conditions of equilibrium of three parallel forces, take any rod, and first find **C** the centre of gravity (*i.e.* the point at which its weight acts) by making it balance on a support placed at that point. Now suspend the rod from two spring balances attached at any two points **A**, **B**, and hanging vertically; at **C** attach any known weight, and read off the spring balances.

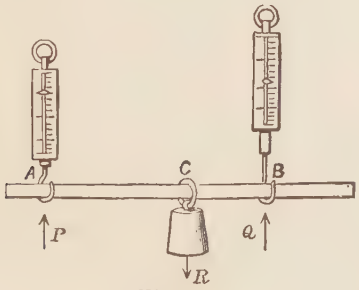


Fig. 77.

(*b*) OBSERVED FACTS.—It will be found that the two readings, when added together, are exactly equal to the weight of the rod together with its attached weight, both of which act at **C**.

Now let the distances **AC**, **CB** be measured. Then, if the experiment be carefully performed, it will be found that the readings of the two balances attached at **A**, **B**, and the total weight at **C**, are proportional, respectively, to the lengths **BC**, **CA**, **AB**.

(*c*) DEDUCTIONS.—Hence, if R denote the total weight at **C**, and P, Q denote the upward thrusts exerted on the rod by the two spring balances, we shall have

$$R = -(P + Q), \quad P \times \mathbf{AC} = Q \times \mathbf{CB},$$

and

$$\frac{P}{\mathbf{BC}} = \frac{Q}{\mathbf{CA}} = \frac{R}{\mathbf{AB}};$$

agreeing with the conditions of equilibrium already found.

Observation.—The experiment can be varied by attaching a third spring balance instead of the weight at **C**. If the three spring balances are held horizontally, the rod resting on a table, the weight of the rod will not affect the equilibrium, and it will be found that the readings of the three spring balances attached to **A**, **B**, **C** are respectively proportional to **BC**, **CA**, **AB**, thus verifying the conditions of § 81.

Summary of Results.

The resultant of two like parallel forces P, Q at **A**, **B** has its

magnitude $= P + Q =$ sum of forces,

direction parallel to P, Q,

point of application **C** *between* **A**, **B**, such that

$$P \times \mathbf{AC} = Q \times \mathbf{CB}. \qquad (\S\ 77.)$$

The resultant of two unequal unlike parallel forces P, Q, where $P > Q$, has its

magnitude $= P - Q =$ difference of forces,

direction parallel to and in sense of greater force P,

point of application **C** on the side of the greater force remote from the lesser (*i.e.* on **BA** produced through **A**), so that

$$P \times \mathbf{CA} = Q \times \mathbf{CB}. \qquad (\S\ 78.)$$

The conditions of equilibrium of three parallel forces P, Q, R acting on a rod at **A**, **B**, **C** may be written

$$\frac{P}{\mathbf{BC}} = \frac{Q}{\mathbf{CA}} = \frac{R}{\mathbf{AB}} \quad \ldots\ldots\ldots\ldots \quad (\mathbf{1}),$$

and the *middle* force is opposite in direction to the other two, and equal to their sum. (§ 81.)

Varignon's Theorem holds good for parallel as well as for intersecting forces, and asserts that the algebraic sum of the moments of two or more forces about any point is equal to the moment of their resultant about that point. (§ 80.)

EXAMPLES VI.

(*Selected Problems.*—1, 2, 3, 5, 7, 8, 9, 10, 11, 12, 13, 15, 17, 18.)

1. Find the resultants of each of the following pairs of like parallel forces at the given distances apart, stating in each case the magnitude of this resultant and its distances from each of the two components, and illustrating by figures :—

(i.) 1 lb. and 3 lb., 2 ft. apart ;
(ii.) 5 lb. and 7 lb., 3 ft. apart ;
(iii.) $10\frac{1}{2}$ lb. and $1\frac{1}{2}$ lb., 1 yd. apart ;
(iv.) 4 lb. and 10 lb., 42 in. apart ;
(v.) $\frac{1}{2}$ ton and $\frac{1}{4}$ ton, 6 in. apart ;
(vi.) 400 and 600 grammes, 10 cm. apart.

2. Find, in like manner, the resultants of pairs of *unlike* parallel forces whose magnitudes and distances apart are given by the data of Example 1.

3. A rod 10 ft. long, whose weight may be neglected, has masses of 8 lb. and 11 lb. attached one to each end. Find the point about which it will balance and the force of pressure on its support.

4. Two unlike parallel forces, of 1 and 2 units respectively, act upon a rigid body, at points 1 ft. apart. Find the magnitude and point of application of their resultant.

5. A uniform bar, 10 ft. long, balances over a rail, with a boy, weighing three times as much as the bar, hanging on to the extreme end of it. Draw a figure showing its balancing position.

6. Show that the resultant of two unlike parallel forces acts towards the side of the greater of the two forces, and can never act between them. What happens if the forces become equal ?

7. Two men are carrying a bar 16 ft. long, and weighing 150 lb. One man supports it at a distance of 2 ft. from one end, and the other man at a distance of 3 ft. from the other end. What weight does each man bear ?

8. A uniform rod, 6 ft. long, and weighing 5 lb., is laid on a table with 6 in. projecting over the edge. What weight can be hung on the end of the rod before the rod will be pulled over ?

9. Two parallel forces P and Q act at two points in a straight line, 6 in. apart, in opposite directions. Their resultant is a force of 1 lb. acting at a point in the line 4 ft. from the larger of the forces P and Q. Determine the values of P and Q.

10. A uniform heavy beam of length 7 ft. rests horizontally on two supports, one at one end and the other $5\frac{1}{2}$ ft. from that end. If the greatest mass that can be hung on the other end of the beam without disturbing the equilibrium be 16 lb., find the weight of the beam.

11. Apply the Equation of Moments to deduce the magnitude and position of the resultant of (i.) two like, (ii.) two unlike, parallel forces applied perpendicular to the straight line joining their points of application.

12. Conversely, employ the results proved in §§ 77, 78 to establish the Equation of Moments for two parallel forces.

13. A rod 12 ft. long, whose weight may be neglected, rests horizontally with one end on the edge of a table and the other supported by a vertical string. A mass of 18 lb. is attached to the rod at a certain point. If the tension of the string be equal to the weight of 12 lb., find the force of pressure on the table and the point where the weight is attached.

14. Find by how much the greater of two parallel forces P, Q acting in opposite directions must be diminished in order that the distance of the line of action of the resultant from that of P may be the same as that of the line of action of the former resultant was from that of Q.

15. **ABC** is a triangle, **O** its orthocentre. A force P acts along **AO**; resolve it into two parallel forces acting at **B** and **C**.

16. In the preceding question, find the result if **O** is the centre of the circumscribed circle.

17. Four forces in the proportion 1 : 2 : 4 : 5 act along the sides of a square taken in order. Prove that their resultant is parallel to a diameter; find where it cuts the side along which the first force acts.

18. A rod, of length $2l$ and weight W, is lying across two pegs on the same level a feet apart. If neither peg can bear a pressure $> P$, find the greatest length of the rod which may project beyond either peg.

CHAPTER VII.

SYSTEMS OF PARALLEL FORCES.—COUPLES.

83. Method of finding the resultant of coplanar parallel forces.

When a number of parallel forces act on a rigid body, it would of course be possible to find their resultant by compounding two of them into a single resultant, then compounding this resultant with a third, and so on. But if the forces all act *in the same plane*, the same thing can be done more easily by writing down the equations which express the facts that—

(i.) *The* **magnitude** *of the resultant equals the algebraic sum of its components* (§ 79) ;

(ii.) *The* **moment** *of the resultant about any point equals the algebraic sum of the moments of its components* (§ 80, Cor. 2).

The point about which moments are taken may be chosen anywhere in the plane of the forces, but some points (very often *one* point) are generally more convenient than others. But it is important to notice that the final result is the same *whatever* point is chosen.

An outside point is generally the best to choose, as there is then no difficulty in distinguishing the signs of the various moments.

As it is advisable to assume *principles* rather than *formulae* in all calculations, we subjoin the following example before giving a general investigation :—

Example.—Weights of 4 lb. and 12 lb. are attached to the ends of a uniform rod, 14 ft. long, weighing 12 lb. To find the point at which their resultant acts.

The three forces on the rod are 4 lb. and 12 lb. acting at its ends, and its weight 12 lb., which, since the rod is uniform, may be supposed to act at its middle point, 7 ft. from either end.

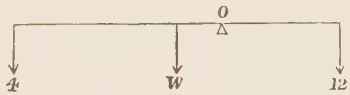


Fig. 78.

The *magnitude* of their resultant = 12 + 12 + 4 lb. = 28 lb.

The required point **O**, at which this resultant cuts the rod, may be found in either of the following ways :—

(*a*) By taking moments about the end at which the 4 lb. acts. Let the distance of **O** from this end be x ft. Then the equation of moments gives $28 \cdot x = 4 \cdot 0 + 12 \cdot 7 + 12 \cdot 14 = 252.$

$$\therefore \quad x = 9 \text{ ft.},$$

or the point **O** is 9 ft. from the weight of 4 lb., and 5 ft. from the other end.

(*b*) By taking moments about the other end. Let the distance of **O** from that end be y. Then

$$28 \cdot y = 12 \cdot 0 + 12 \cdot 7 + 4 \cdot 14 = 140.$$

$$\therefore \quad y = 5 \text{ ft.},$$

or the point **O** is 5 ft. from the 12 lb. weight, agreeing with (*a*).

(*c*) By taking moments about the middle point. Let the distance of **O** from the middle point be z. Then

$$28 \cdot z = 12 \cdot 7 + 12 \cdot 0 - 4 \cdot 7 = 56.$$

$$\therefore \quad z = 2 \text{ ft.}$$

Hence the distances of **O** from the ends are 7 + 2 ft. and 7 − 2 ft.; *i.e.* 9 and 5 ft., agreeing with (*a*) and (*b*).

(*d*) By taking moments *about* **O** *itself*. Let its distance from the middle point be z as before. Since the resultant passes through **O**, it has no moment about **O**. Therefore

$$0 = 12 \cdot (7 - z) - 12 \cdot z - 4 \cdot (7 + z)$$

$$= 84 - 12z - 12z - 28 - 4z = 56 - 28z;$$

giving $z = 2$ ft.,

agreeing with (*c*). This last method does not require us to first find the magnitude of the resultant, but no *material* advantage is thereby gained.

84. To find the resultant of any number of given parallel forces P_1, P_2, P_3, ... acting in one plane.

Let P_1, P_2, P_3, ... denote the given forces both in magnitude and algebraic sign. Draw any line $OA_1A_2A_3$ ... at right angles to the forces. On it take any point **O**, the distances OA_1, OA_2, ... being known.

(i.) To find the magnitude of the resultant.

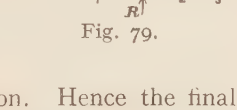


Fig. 79.

The resultant of P_1, P_2 is a parallel force of magnitude $P_1 + P_2$; the resultant of this force and P_3 is therefore $P_1 + P_2 + P_3$; and so on. Hence the final resultant is a parallel force R, such that

$$R = P_1 + P_2 + P_3 + \ldots \ldots \ldots \ldots \ldots (1).$$

= algebraic sum of the forces.

(ii.) To find the position of the resultant R.

Let it cut OA_1 ... in a point **G**, the position of which is required. Since the moment of the resultant is equal to the algebraic sum of the moments of the components,

$$\therefore \quad R \,.\, OG = P_1 \,.\, OA_1 + P_2 \,.\, OA_2 + P_3 \,.\, OA_3 + \ldots .$$

Hence $$OG = \frac{P_1 \,.\, OA_1 + P_2 \,.\, OA_2 + P_3 \,.\, OA_3 + \ldots}{R},$$

or $$OG = \frac{P_1 \,.\, OA_1 + P_2 \,.\, OA_2 + P_3 \,.\, OA_3 + \ldots}{P_1 + P_2 + P_3 + \ldots} \quad \ldots (2).$$

Equations (1), (2) determine, respectively, the magnitude and position of the resultant.

If x_1, x_2, x_3, . . denote the known distances of the component forces from **O**, and $\bar{x}$ the required distance of their resultant; equation (2) becomes

$$\bar{x} = \frac{P_1x_1 + P_2x_2 + P_3x_3 + \cdots}{P_1 + P_2 + P_3 + \ldots} \quad \ldots \ldots \ldots (2a).$$

85. Equilibrium of a loaded beam resting on two supports. When a beam is loaded with given weights placed at given points, and rests in a horizontal position on two props, it is often necessary to determine the forces of pressure on the props, or, what amounts to the same thing, the reactions of the props on the rod (which are equal and opposite to them).

These reactions, together with the weights on the beam, form a system in equilibrium, and therefore

(i.) The sum of the forces is zero,

(ii.) The sum of their moments about any point is zero.

If we want to find the reactions one at a time, we therefore proceed as follows :—

(i.) *Take moments about one of the props. The reaction of that prop has no moment, and therefore the equation of moments at once gives the reaction of the other prop.*

(ii.) *Equate to zero the algebraic sum of the forces (including the two reactions). The equation gives the sum of the reactions, and hence the other required reaction.*

Thus let a beam, whose weight W acts at its centre of gravity **G**, be supported at **M, N**, and let any weights w_1, w_2, w_3 be attached at points $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$. Then, if R, S denote the unknown reactions at **M, N**, the equation of moments about **N** gives

$$R \,.\, \mathbf{NM} = w_1 \,.\, \mathbf{NA}_1 + w_2 \,.\, \mathbf{NA}_2 + w_3 \,.\, \mathbf{NA}_3 + W \,.\, \mathbf{NG} ;$$

and the equation of moments about **M** gives

$$S \,.\, \mathbf{MN} = w_1 \,.\, \mathbf{MA}_1 + w_2 \,.\, \mathbf{MA}_2 + w_3 \,.\, \mathbf{MA}_3 + W \,.\, \mathbf{MG}.$$

By addition, $(R+S)\ \mathbf{MN} = w_1.\mathbf{MN} + w_2.\mathbf{MN} + w_3.\mathbf{MN} + W.\mathbf{MN}$,

or $$R + S = w_1 + w_2 + w_3 + W.$$

This is the equation which we should obtain by resolving perpendicular to the rod, or equating the algebraic sum of the forces to zero, showing that the values of R, S found by resolving and taking moments about the supports are consistent.

If, in any example, the thrust on one of the props should come out *negative*, it is to be inferred that the rod presses *upwards* on its support, and that the latter has to hold it down.

Examples.—(1) A uniform rod, 10 ft. long, of weight W, is supported on trestles at both ends, and a weight $5W$ is placed on it 4 ft. from one end. To find the forces of pressure on the trestles.

Let **AB** be the rod, **C** the middle point at which its weight W may be supposed to act, **D** the point of application of the weight $5W$.

Let R, S be the required forces of pressure at **A, B.**

Then, by equating forces, we have

$$R + S = 5W + W = 6W \ldots\ldots\ldots\ldots \text{(i.)}$$

Taking moments about **A**, we have

$$S \times 10 = W \times 5 + 5W \times 4 \quad \text{.....(ii.)}$$

From (ii.), $S = \frac{5}{2}W$. Hence, by (i.), $R = \frac{7}{2}W$.

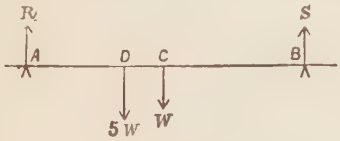


Fig. 80.

(2) A uniform rod, 12 ft. long, weighing 20 lb., has weights of 12 lb. and 4 lb. attached to its ends, and 8 lb. attached at a distance of 4 ft. from the 4-lb. weight. It is placed on two props, 8 ft. apart, so that the end with the 4-lb. weight projects 1 ft. To find the reactions of the props.

In Fig. 81, let **M, N** be the props, and let their reactions be R lb. and S lb., respectively.

Let **G** be the middle point of the rod at which its weight (20 lb.) acts.

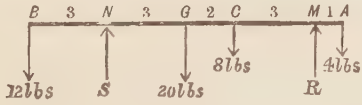


Fig. 81.

(i.) Taking moments about **M**, we have

$$S \,.\, \mathbf{MN} = 12 \,.\, \mathbf{MB} + 20 \,.\, \mathbf{MG} + 8 \,.\, \mathbf{MC} - 4 \,.\, \mathbf{AM},$$

or
$$8S = 12 \,.\, 11 + 20 \,.\, 5 + 8 \,.\, 3 - 4 \,.\, 1$$
$$= 132 + 100 + 24 - 4 = 252;$$

whence
$$S = 31\tfrac{1}{2} \text{ lb.}$$

(ii.) Again, since the sum of the upward forces is equal and opposite to the sum of the downward ones (or the algebraic sum of the forces is zero),

$$\therefore \quad R + S = 12 + 20 + 8 + 4 = 44;$$

whence
$$R = 44 - 31\tfrac{1}{2} \text{ lb.} = 12\tfrac{1}{2} \text{ lb.}$$

Hence the thrusts on the props are $12\frac{1}{2}$ lb. and $31\frac{1}{2}$ lb.

[If we had taken moments about **N**, we should have had

$$R \,.\, \mathbf{NM} = 4 \,.\, \mathbf{NA} + 8 \,.\, \mathbf{NC} + 20 \,.\, \mathbf{NG} - 12 \,.\, \mathbf{BN},$$

$$8R = 4 \,.\, 9 + 8 \,.\, 5 + 3 \,.\, 20 - 12 \,.\, 3$$

$$= 36 + 40 + 60 - 36 = 100\,;$$

whence $R = 12\frac{1}{2}$ lb.,

agreeing with the value just found and affording a test of the accuracy of the calculation.]

86. DEFINITION.—**A couple** consists of two equal forces acting in opposite directions along two parallel straight lines. *A couple cannot keep a body in equilibrium*, for it tends to rotate the body, the points of application of the two forces of the couple tending to move in opposite directions (§ 51). Moreover, the proof that two parallel forces have a single resultant fails for the case of a couple (§ 75).

Examples of couples.—In winding a clock we apply a *couple* to the key, for we do not try to make it move to one side or the other, but simply turn it round. To spin a small top between the finger and thumb, we apply a *couple* to it by moving the finger and thumb sharply in opposite directions. To open a door we apply a *couple* to the handle.

87. **Resultant of two parallel forces which nearly form a couple.**—Consider the resultant of two unlike parallel forces which are very nearly, but not quite, equal. Let P, Q be two such forces acting at **A**, **B**, and let $P > Q$. If R be the resultant of P and Q,

$$R = P - Q\,;$$

therefore, if Q is nearly equal to P, R is very small.

Also, if **C** be the point where the resultant meets **AB** produced,

$$\mathbf{AC} \times R = \mathbf{BA} \times Q.$$

$$\therefore \quad \mathbf{BA} = \mathbf{AC} \times \frac{R}{Q}$$

$$= \mathbf{AC} \times \frac{P - Q}{Q}.$$

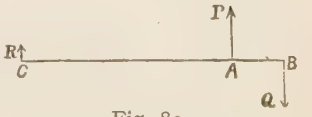


Fig. 82.

Since $P - Q$ is small, therefore **BA** is small in comparison with **AC**; hence **AC** is large in comparison with **BA**. Hence the resultant acts at a great distance from the line of action of either force. Writing the last equation,

$$\mathbf{AC} = \mathbf{BA} \times \frac{Q}{P - Q},$$

we see that, if $Q = P$, the denominator becomes zero, and **AC** becomes infinitely great.

Hence *when two unlike parallel forces approach equality, and finally become equal, their resultant becomes infinitely small, and its line of action moves to an infinitely great distance from the components.*

For this reason it is convenient to consider the properties of couples apart from those of other systems of forces.

88. DEFINITIONS.—The **arm** of a couple is the perpendicular distance (**AB**, Fig. 83) between the lines of action of its two components (*i.e.* the two forces forming the couple).

The **moment** of a couple is the algebraic sum of the moments of its two components about any point in their plane.

The following is the fundamental property of couples :—

89. The moment of a couple is the same about all points in its plane.

Let the couple consist of two equal and opposite forces P, $-P$ at **B** and **A**. Let **O** be any point in their plane. Draw **OAB** perpendicular to the forces.

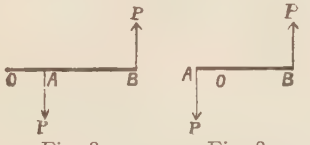


Fig. 83. Fig. 84.

Then, if **O** does not lie between **A** and **B**, as in Fig. 83, we have

algebraic sum of moments of forces $= P \,.\, \mathbf{OB} - P \,.\, \mathbf{OA}$

$= P\,(\mathbf{OB} - \mathbf{OA}) = \boldsymbol{P} \,.\, \mathbf{AB}.$

If **O** lies between **A** and **B**, as in Fig. 84, we have

algebraic sum of moments $= P \,.\, \mathbf{OB} + P \,.\, \mathbf{AO}$

$= P\,(\mathbf{AO} + \mathbf{OB}) = \boldsymbol{P} \,.\, \mathbf{AB}.$

Hence the moment of the couple about **O** *is independent of the position of* **O** *and is equal to the product* $\boldsymbol{P}$ **. AB.**

90. Alternative expressions for the moment of a couple.

The **moment** of a couple may therefore be defined as—

(i.) *The product of the measure of either force into the arm of the couple.*

(ii.) *The moment of either of the two forces about any point in the line of action of the other force.* For

moment $= P \times \mathbf{AB}$ = moment about **A** of P acting at **B**.

91. **A couple cannot be replaced by a single force.**

For the moment of a single force about any point on its line of action is zero (§ 61). But the moment of a couple about every point in its plane is a constant quantity, differing from zero.

Hence a couple cannot have a resultant. The same thing also follows from § 87.

92. A force, acting at any part of a body, is equivalent to an equal and parallel force acting at any other point together with a couple.

Let P be a given force acting at any point **O**; to show that it is equivalent to an equal and parallel force P, acting at any other point **A**, together with a couple.

Introduce two equal and opposite forces P, $-P$, acting at **A**, numerically equal and parallel to the force **P** at **O**.

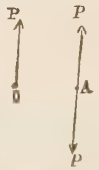


Fig. 85.

The effect of these forces will be to neutralise one another.

But the forces P at **O** and $-P$ at **A** form a couple whose moment (about **A**) is equal to the moment of the original force P about **A**.

Thus the original force P at **O** is equivalent to this couple and a parallel and equal force P at **A**.

93. Two couples in the same plane whose moments are equal and opposite will balance one another.

Let one of the couples consist of two forces $P, -P$, acting on the arm **AB**, and let the other couple consist of forces $Q, -Q$, acting on the arm **CD**.

Suppose, firstly, that P and Q are not parallel. Then the lines of action of the four forces must form a parallelogram **abcd**.

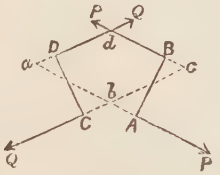


Fig. 86.

Since the moments of the couples are equal and opposite about any point in the plane, therefore the moments about **b** of P and Q acting along **ad** and **cd** are equal and opposite, and therefore the moment of the resultant of these forces about **b** is zero (§ 69).

Therefore the resultant of P and Q at **d** must pass through **b**, and therefore it acts along **bd**.

Similarly the resultant of $-P, -Q$ at **b** acts along **db**.

But the latter resultant is equal and opposite to the former, for the two components of the latter are respectively equal and opposite to those of the former.

Hence the two resultants balance each other, and therefore the four forces forming the two couples are in equilibrium.

To extend the proof to the case where the forces composing the two couples are parallel, it is only necessary to introduce two equal and opposite forces in a straight line, intersecting the forces $P, -P$ of one of the couples, and to compound one of these forces with P and the other with $-P$. We thus get a couple equivalent to the original one and of equal moment, but its component forces intersect those of the second couple; hence the above proof applies.

94. A couple acting on a rigid body may be replaced by any other couple of equal moment acting on the body in the same plane without altering its effect.

Let M be the moment of the given couple. Apply two couples of equal and opposite moments $M, -M$ to the body *anywhere* in the plane of the first couple. These balance each other, and do not therefore affect the body. Now combine the *first* and *third* couples. These also balance each other, since their moments $(M, -M)$ are equal and opposite; therefore they may be removed. We are thus left with the *second* couple of moment M as the equivalent of the *first*.

Cor. From this result we see that a couple has no particular position of application, but that it may be shifted anywhere in its plane without altering its statical effect. The effect of the couple depends therefore only on its moment and the plane in which it acts.

Observation.—The above proof is identical in its reasoning with the proof of the Principle of Transmission of Force. The present theorem may therefore be called the *Principle of the Transmission (or Transmissibility) of Couples.*

***95. To compound two coplanar couples into a single resultant couple.**—Let the couples consist of the forces $(P, -P)$ and $(Q, -Q)$ (Fig. 86), and let their lines of action be produced, if necessary, so as to form a parallelogram **abcd**.

Let R be the resultant of P and Q acting at **d**.

Then the resultant of $-P$, $-Q$ acting at **b** is evidently an equal and opposite force $-R$, parallel to R.

The forces R at **d** and $-R$ at **b** form a couple which is the required resultant of the two couples $(P, -P)$ and $(Q, -Q)$.

Since R is the resultant of P, Q acting at **d**,

$\therefore$ moment of R about **b**

= algebraic sum of moments of P, Q about **b**.

But **b** is a point on the line of action of the forces $-P$, Q, $-R$.

Therefore the moments about b are the moments of the respective couples, and

$\therefore$ *moment of resultant couple*

$=$ *algebraic sum of moments of its component couples.*

Cor. *Any number of coplanar couples are equivalent to a single couple whose moment is the algebraic sum of their moments.* [For compound the couples together two at a time, etc.]

96. **Any number of coplanar forces are equivalent either to a single force or to a couple, or are in equilibrium.**

Any two forces not forming a couple can be compounded together; and three or more forces cannot each form couples with all the others, for they cannot all be unlike. Therefore two of them must be capable of being replaced by a single force, and this process can be repeated until there are only two forces left. These either have a single resultant or form a couple or balance.

Examples.—(1) **ABCD** is a rectangle whose sides **AB, BC** measure 8 and 3 inches respectively (Fig. 86*a*). Forces of 16, 5, 16, 9 lb. wt. act along the sides **BA, BC, DC, DA** respectively. Find their resultant.

The forces along **BA** and **DC** form a couple whose moment is 16×3 or 48, in inch-lb. units of moment. This couple can be replaced by a couple consisting of two forces 6, 6 lb. wt. acting along **AD, CB** respectively (§ 94); for the moment of this couple would be 6×8 or 48. We now have forces 6 along **AD**, 9 along **DA** (which reduce to 3 along **DA**); and also 6 along **CB**, 5 along **BC** (which reduce to 1 along **CB**).

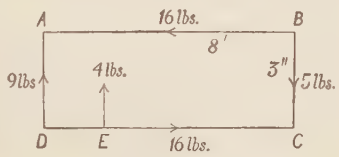


Fig. 86*a*.

Thus the whole system is reduced to 3 along **DA** and 1 along **CB**. These two forces can be compounded by § 77, and give a resultant of magnitude 4 lb. wt. acting at a point **E** 2 inches to the right of **D**, as represented in the figure.

(2) Show that the system of forces represented in the accompanying diagram (Fig. 86*b*) are equivalent to a couple, and determine its moment.

The resultant of the two upward forces is 8 lb. wt. acting vertically up, and the resultant of the two downward forces is 8 lb. wt. acting vertically down. These two forces of 8 lb. wt. will be in equilibrium or form a couple according as they do or do not act in the same line.

Hence the given system is either in equilibrium or equivalent to a couple.

If it is in equilibrium the algebraic sum of moments of the system about any point is zero. If it is equivalent to a couple the algebraic sum of moments of the system about any point is equal to the moment of the couple. (Both these statements are easily deduced from § 80.)

But the algebraic sum of moments of the system about **D** is

$$-1\times\mathbf{AD}+3\times\mathbf{BD}-7\times\mathbf{CD}+5\times 0,$$

i.e. $-4+9-7+0$ or -2 ft.-lb. units. Hence the system must be equivalent to a couple of moment -2.

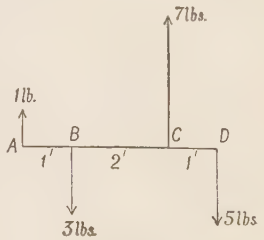


Fig. 86*b*.

Summary of Results.

If a number of parallel forces P_1, P_2, P_3, etc., cut a straight line at distances x_1, x_2, x_3, etc., from any point **O**, their resultant is given by the equation

$$R = P_1 + P_2 + P_3 + \ldots\ldots\ldots (1),$$

and its distance $\bar{x}$ from **O** by

$$\bar{x} = \frac{P_1x_1 + P_2x_2 + P_3x_3 + \ldots}{P_1 + P_2 + P_3 + \ldots} \quad \ldots\ldots (2),$$

provided the proper sign is given to each force and distance. (§ 84.)

Couple = two equal unlike parallel forces. (§ 86.)

Moment of couple

= algebraic sum of moments of its forces about any point in its plane (§§ 88, 89.)

= moment of one force about point on the other (§ 90.)

= P . **AB** (where P = either force, **AB** = arm). (§ 90.)

Principle of transmission of couples.—A couple may be replaced by any other couple of equal moment. (§ 94.)

Resultant of two or more couples is a couple whose moment = sum of moments of components. (§ 95.)

EXAMPLES VII.

(*Selected Problems* :—1, 4, 7, 8, 11, 12, 15, 17, 18, 19, 20, 23, 24, 25.)

1. Two men carry a load of $\frac{3}{4}$ cwt. suspended from a horizontal pole 14 ft. long, whose weight is 10 lb., and whose ends rest on their shoulders. Find the point at which the load must be suspended in order that one of the men may bear 71 lb. of the whole weight.

2. Three like parallel forces of 5 lb., 7 lb., and 9 lb. act in lines whose distances apart are 3 ft. and 4 ft. Find the resultant.

3. A uniform beam, 14 ft. long and weighing 120 lb., is attached to two props, one of which is 3 ft. and the other 5 ft. from its centre. Calculate the forces on the props when a weight of 100 lb. is placed first at one end and then at the other end of the beam.

4. A uniform bar, 3 ft. long and weighing 5 lb., rests on a horizontal table with one end projecting 4 in. over the edge. Find the greatest weight that can be hung on the end without making the bar topple over.

5. If a beam 6 ft. long, and weighing 15 lb., is acted upon by a downward force of 3 lb. at one end and an upward force of 7 lb. at the other end, what force is required to keep it in equilibrium in a horizontal position?

6. A bar of uniform thickness and density, 12 ft. long and weighing 1 cwt., is supported at its extremities in a horizontal position. If a body of 2 cwt. be suspended from a point 2 ft. distant from one end, and a body of 4 cwt. at 4 ft. from the other end, required the pressures on the points of support

7. A uniform beam, 12 ft. long and weighing 72 lb., is supported on two props, 1 ft. and 2 ft., respectively, from the ends. Where must a weight of 24 lb. be hung to make the pressure on each prop equal to 48 lb.?

8. A rod without weight rests horizontally on two points, **A** and **B**, 10 ft. apart. Between **A** and **B** take points **C**, **O**, **D**, such that **AC** = 2 ft., **AO** = 4 ft., **AD** = 7 ft. A weight of 100 lb. is hung at **C**, and one of 90 lb. at **D**. Find the algebraical sum of the moments with respect to **O** of the forces on one side of **O**.

9. A uniform beam, 12 ft. long and weighing 56 lb., rests on, and is fastened to, two props 5 ft. apart, one of which is 3 ft. from one end of the beam. A load of 35 lb. is placed (*a*) on the middle of the beam, (*b*) at the end nearest a prop, (*c*) at the end furthest from a prop. Calculate the weight each prop has to bear in each case.

10. A rod is supported horizontally on two points **A** and **B**, 12 ft. apart. Between **A** and **B** points **C** and **D** are taken, such that **AC** = **BD** = 3 ft. A weight of 120 lb. is hung at **C**, and a weight of 240 lb. at **D**; the weight of the rod is neglected. Take a point **O** midway between **A** and **B**, and find with respect to it the algebraical sum of the moments of the forces acting on the rod on one side of **O**.

11. When a force and a couple act in the same plane on a rigid body, find their resultant.

12. Draw a square **ABCD** and its diagonal **AC**. Two forces of 10 units act from **A** to **B** and from **C** to **D** respectively, forming a couple; a third force of 15 units acts from **C** to **A**. Find their resultant, and show in a diagram exactly how it acts.

13. Prove that a couple can be moved parallel to itself without altering its effect.

14. State (without proof) the conditions that must be satisfied in order that two couples may balance. Give a practical illustration of two balancing couples in different planes.

15. Forces P, Q, R act along the sides of a triangle **ABC** from **B** to **C**, **C** to **A**, and **A** to **B**, and are proportional to the lengths of the sides along which they act. (*a*) Show that they form a couple, and find its moment; (*b*) find their resultant when the direction of one of them (P) is reversed.

16. Forces P and Q act at **A**, and are completely represented by **AB** and **AC**, sides of a triangle **ABC**. Find a third force R, such that the three forces together may be equivalent to a couple whose moment is represented by half the area of the triangle.

17. Express the work done when a moment M has rotated n times. If a force equal to the weight of 10 lb. revolve three times tangentially round a circle of 5 ft. radius, find the work it would do.

18. Forces act along the sides of a polygon, and are represented completely by those sides taken in order. Show that they are equivalent to a couple whose moment is measured by twice the area of the polygon.

19. Three forces, acting along the sides of a triangle taken in order, are equivalent to a couple; prove that they are proportional to the sides of the triangle.

20. Four forces acting along the sides of a parallelogram are equivalent to a couple. Prove that the forces along opposite sides are equal in magnitude and opposite in sense.

21. Three forces, acting at the angular points of a triangle **ABC** along the tangents to the circumscribed circle, are equivalent to a couple. Find their ratios.

22. Two equal unlike parallel forces, acting at fixed points **A** and **B**, form a couple of moment G; if their lines of action are turned through a right angle, they form a couple of moment H. Show that, when they both act at right angles to **AB**, they form a couple of moment $\sqrt{(G^2 + H^2)}$.

23. Three parallel forces P, Q, R, acting at the angular points of the triangle **ABC**, are in equilibrium when they act at right angles to **BC**. Prove that if each line of action be turned through an angle α, in the same direction, they will be equivalent to a couple of moment

$$Pb \sin \alpha \,.\, \sin C.$$

24. Three forces, proportional to the sides of a traingle, act perpendicularly to these sides. Prove that they are either in equilibrium or form a couple.

25. **ABC** is an equilateral triangle of side a; **D**, **E**, **F** divide the sides **BC**, **CA**, **AB** respectively in the ratio $n : 1$. Three forces, each equal to P, act at **D**, **E**, **F**, perpendicularly to the sides, and outward from the triangle. Prove that they are equivalent to a couple of moment

$$\frac{3\,(n-1)}{2\,(n+1)}\,Pa.$$

26. In the quadrilateral **ABCD**, **AB** and **CD** are parallel but not equal; **AD** and **BC** are equal but not parallel. If forces P, Q, R, S, acting along the sides taken in order (P acting along **AB**), are equivalent to a couple, prove that

$$Q = S = \tfrac{1}{2}(R - P) \sec \mathbf{ADC}.$$

27. If, in the preceding problem, **AD** and **BC** are neither equal nor parallel, prove that

$$Q \sin \mathbf{B} = S \sin \mathbf{D}, \quad \text{and} \quad Q \cos \mathbf{B} + S \cos \mathbf{D} = R - P.$$

EXAMINATION PAPER III.

1. What do you mean by the *moment* of a force about a point?

2. Show that the algebraical sum of the moments of two forces (whose lines of action intersect) about any point in the plane containing the forces is equal to the moment of their resultant.

Deduce the rule for compounding two like parallel forces.

3. Find the resultant of two parallel forces acting in opposite directions.

4. **ABC** is a triangle having a right angle at **C**; **BC** is 12 ft., and **AC** is 20 ft.; **P** is a point in the hypotenuse **AB** such that **AP** is one-fourth of **AB**; a force of 50 lb. acts from **C** to **B**, and one of 100 lb. from **C** to **A**. (*a*) Find the moments of the forces with respect to **P**. (*b*) Find the sum (*i.e.* the algebraical sum) of the two moments. (*c*) If the point **P** were fixed, in what direction would the forces make the triangle revolve?

5. State the conditions of equilibrium of three parallel forces acting upon a rigid body.

6. A straight line **AB** represents a rod, 10 ft. long, supported horizontally on two points, one under each end; **C** is a point in **AB**, 3 ft. from **A**. What thrust is produced on the points **A** and **B** by a weight of 30 lb. hung at **C**? What additional thrust is exerted on the points of support if the rod is uniform and weighs 20 lb.?

7. Explain why a rod which will support a considerable tension can be broken comparatively easily by being bent.

8. What is the resultant of a force of 4 lb. weight and a couple with an arm 2 ft. long and forces 3 lb. weight? State its position clearly with reference to the 4 lb. force.

9. What is meant by a *couple* in Mechanics? Find the condition that two couples which act on a rigid body should equilibriate.

10. Show that any number of couples applied to a rigid body in one plane are equivalent to a single couple whose moment is the algebraic sum of the moments of the individual couples.

CHAPTER VIII.

MACHINES—THE LEVER—THE WHEEL AND AXLE.

97. **A machine** in Mechanics means any contrivance in which a force applied at one point is made to raise a weight or overcome a resisting force acting at another point. The former force is called the **effort** or **power**, the latter the **resistance** or **weight.***

In what follows, the effort will be denoted by P, and the resistance by either Q or W, the letter W being generally used when the resistance is a heavy weight which the machine has to lift.

Machines are used for the following purposes :—

(1) To enable a person to raise weights or overcome resistances so great that the effort he is capable of exerting would be insufficient without the use of a machine.

Example.—A truck drawn up an inclined plane to any required height when it is too heavy to be lifted bodily off the ground.

(2) To enable the motion imparted to one point of a machine to produce a much more rapid motion at some other point.

Example.—A bicycle, or a winnowing machine.

(3) To enable the effort to be applied at a more con-

* The terms " power " and " weight " are used in the older books on Mechanics, and still sometimes occur in examination papers; but " power " also signifies " rate of working," such as horse-power, and machines are often used in overcoming resistances other than those due to gravity or " weight."

venient point, or in a more convenient direction, than that in which the resistance acts.

Example.—A poker used to stir the fire, a rope and pulley for raising a bucket from a well.

98. **The mechanical powers.**—The simplest forms of machines are called the **mechanical powers,** and it is usual to distinguish the following six forms of them :—

The inclined plane. [Chap. III.]
The wedge. [Chap. IV.]
The lever.
The wheel and axle, and windlass.
The pulley and systems of pulleys. [Chap. IX.]
The screw. [Chap. IX.]

In every case we suppose these machines to be devoid of friction, and in Statics we are chiefly concerned with finding the relations between the effort and the resistance, when there is equilibrium.

It may be thought that the subject of machines should belong to Dynamics rather than Statics, as the object of any machine is to produce motion, not equilibrium. For example, a workman often uses a system of pulleys for lifting a heavy weight.

If, however, we calculate the force which the workman would have to exert in order to keep the heavy weight suspended, any slight extra force will produce a slow motion, and a slow motion is all that is required.

99. **Mechanical advantage.**—DEFINITION.—The **mechanical advantage** is the number which expresses what multiple the resistance is of the effort, *i.e.*

$$\text{mechanical advantage} = \frac{\text{resistance}}{\text{effort}} = \frac{Q}{P} \text{ or } \frac{W}{P}.$$

Consider, for example, a smooth inclined plane at a slope of, say, 1 in 20. By applying a force of 1 cwt. along the plane, it is possible to draw a weight of 20 cwt. or 1 ton up the plane, and if the plane be long enough, this weight may be raised to any desired height. The resistance to be overcome is that due to gravity, viz. the weight of 20 cwt. It is therefore twenty times the effort, and we say that *the mechanical advantage is* 20. Generally, if the effort acts up a smooth incline of 1 in n, the mechanical advantage is n.

100. The lever is a rigid bar capable of turning freely about a fixed point of support. This point is called the **fulcrum.** The *effort* is a force applied at any point of the lever, so as to turn it about the fulcrum, and thus to raise a weight or overcome a resistance applied at any other point.

The lever is most often a straight rod, the two arms therefore being in one straight line, and the effort and resistance generally act perpendicular to the arms. But these are mere matters of convenience.

In theoretical calculations we neglect the thickness of the lever and most frequently assume it to be without weight.

101. To find the mechanical advantage of the lever when the forces act perpendicular to the arms.—Let **C** be the fulcrum, **CA, CB** the arms, and let a force P applied at **A** perpendicular to **CA** support a resistance Q applied at **B** perpendicular to **CB.** Then the condition of equilibrium

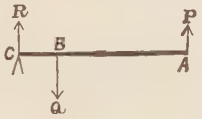


Fig. 87.

requires the moments of P and Q about **C** to be equal and opposite, and therefore

$$P \times \mathbf{CA} = Q \times \mathbf{CB}.$$

This condition may be written

$$\frac{Q}{P} = \frac{\mathbf{CA}}{\mathbf{CB}}.$$

$$\therefore \text{ mechanical advantage } \frac{Q}{P} = \frac{\text{arm of effort}}{\text{arm of resistance}} \quad (1);$$

or *the effort and resistance are inversely proportional to the arms on which they act.*

Hence, by applying the effort at the end of a long arm and the resistance very near the fulcrum, the mechanical advantage may be made very great, and a man may raise a weight many times greater than he could lift bodily off the ground.

Example.—Thus, by exerting a force of 1 lb. at a distance of 2 ft. from the fulcrum, we can lift a weight of 2 lb. applied at a distance of 1 ft. from the fulcrum, or a weight of 24 lb. applied an inch away from the fulcrum.

102. **Mechanical advantage of levers in general.**—Where P, Q do not act perpendicular to **CA, CB**, we must drop **Ca, Cb** perpendicular on their lines of action.

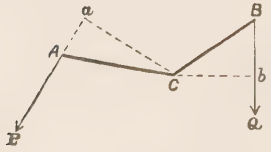


Fig. 88.

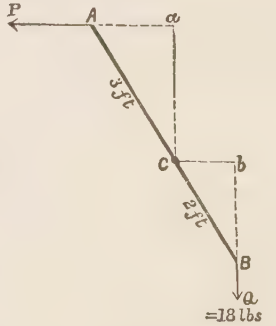


Fig. 89.

Taking moments about **C**, the condition of equilibrium now becomes $P \times \mathbf{Ca} = Q \times \mathbf{Cb}$.

$$\therefore \text{ mechanical advantage } \frac{Q}{P} = \frac{\mathbf{Ca}}{\mathbf{Cb}}$$

$$= \frac{\text{perp. dist. of effort from fulcrum}}{\text{perp. dist. of resistance from fulcrum}}.$$

Observations.—The only difference therefore is that the perpendiculars Ca, Cb are no longer the arms of the lever. We might, if

we liked, replace the lever with one whose arms are **Ca, Cb** without altering the mechanical advantage.

As a rule, in all such cases it is advisable to start with the Equation of Moments in working numerical examples, and not to assume formulae which are only particular cases of that equation.

Example.—A straight lever, whose arms are 2 ft. and 3 ft. long, rests at an inclination of 60° to the horizon, and a weight of 18 lb. hangs vertically from its shorter arm. To find the horizontal force which must be applied to its longer arm in order to balance.

Let **ACB** be the lever (Fig. 89). Then, if P, Q denote the effort and weight, these forces make angles of 60° and 30° respectively with **AB**; therefore their resolved parts perpendicular to **AB** are $\frac{1}{2}P\sqrt{3}$ and $\frac{1}{2}Q$ respectively. Their moments about **C** are the products of these resolved parts into the arms **CA, CB**. Therefore the Equation of Moments gives

$$\frac{1}{2}P\sqrt{3} \times \mathbf{CA} = \frac{1}{2}Q \times \mathbf{CB};$$

$$\therefore \quad \frac{1}{2}P\sqrt{3} \times 3 = \frac{1}{2} \cdot 18 \times 2;$$

whence required force $P = \dfrac{18 \times 2}{3\sqrt{3}}$

$$= \frac{18 \times 2\sqrt{3}}{9} = 4\sqrt{3} \text{ lb.}$$

103. **The three classes of lever.**—*Straight* levers are sometimes divided into three classes, according to the relative positions of **A, B, C,** the points of application of the effort and resistance and the fulcrum. We shall suppose the effort and resistance to be parallel. In considering the different cases, it is convenient to use the symmetrical conditions of equilibrium of three parallel forces. If R be the reaction of the fulcrum on the lever, we have, therefore, by § 81,

$$\frac{P}{\mathbf{BC}} = \frac{Q}{\mathbf{CA}} = \frac{R}{\mathbf{AB}},$$

the *middle* force acting in the opposite direction to the outer ones and being equal to their sum.

The thrust of the lever against its support at **C** is a force equal and opposite to R.

For convenience we shall often suppose that the resistance Q is a weight which acts downwards.

104. A lever of the first class (Fig. 90) is one in which the fulcrum is placed between the effort and the resistance.

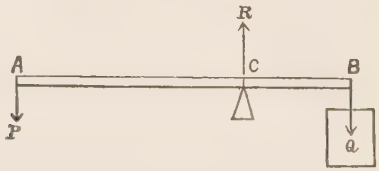


Fig. 90.

Here R is the middle force; therefore P, Q act in the same direction, and R in the opposite direction, also

$$R = P + Q.$$

Thus, in order to lift a weight, the effort must be applied *downwards*, and the reaction acts *upwards* (so that the lever presses *downwards* on the fulcrum).

In this lever, **BC** may either be greater or less than or equal to **CA.** Therefore the effort may either be greater or less than or equal to the weight which it has to lift; so that *the mechanical advantage may be less or greater than, or equal to unity.*

Examples of this class.—The handle of a pump; a crow-bar when it rests on a block in front of the weight to be

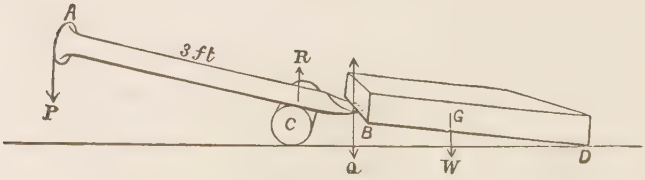


Fig. 91.

lifted and not with its end on the ground (Fig. 91); a poker, used to raise the coals in the grate, a bar of which is the fulcrum; a spade, in digging; a see-saw.

Double lever.—A pair of scissors. Fig. 92 shows the forces acting on the arms of the scissors those on the dotted arm being accented.

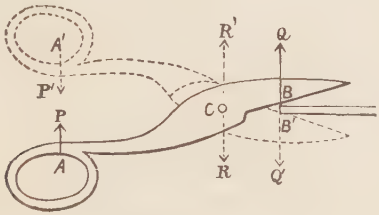


Fig. 92.

105. A lever of the second class (Fig. 93) is one in which the resistance is placed between the fulcrum and the effort.

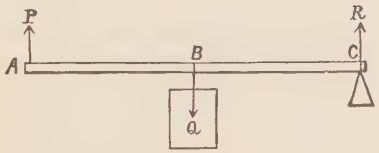


Fig. 93.

Here Q is the middle force; therefore P and R act in the same direction, and Q in the opposite direction, also

$$Q = P + R, \quad R = Q - P,$$

Thus, in order to lift a weight, the effort must be applied *upwards*, and the reaction of the fulcrum also acts *upwards* (so that the lever presses *downwards* on the fulcrum).

Since **CA > CB**, the effort is less than the weight; so that *the mechanical advantage is always greater than unity.*

Examples of this class.—A wheelbarrow (Fig. 94), the fulcrum being where the wheel touches the ground; a crowbar, when the lower end rests on the ground.

NOTE.—In the wheelbarrow half the effort is applied at each handle.

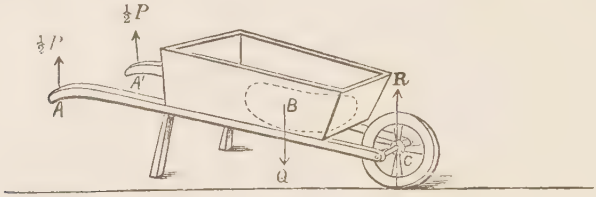


Fig. 94.

Double lever.—A pair of nut-crackers, the forces on the two arms being shown in Fig. 95.

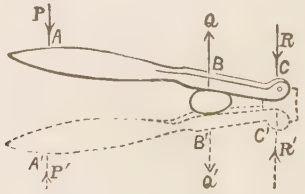


Fig. 95.

An oar is often called a lever of the second class. It cannot be strictly said to belong to either class. If the boat were kept at rest and the oar used to scoop the water backwards, it would be a lever of the first class, with the rowlock as fulcrum. When the boat moves forwards instead of the water moving backwards, the relative motion and the relation between the effort applied to the handle and the resistance of the water are the same as before, *and can be correctly found by treating the oar as a lever of the first class.**

* This is proved in a note published in the *Philosophical Magazine* for Jan., 1887.

106. A lever of the third class (Fig. 96) is one in which the effort is applied between the fulcrum and the resistance.

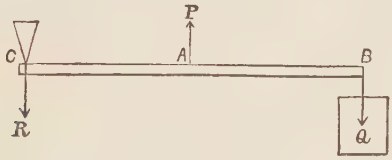


Fig. 96.

Here P is the middle force; therefore Q, R act in the same direction, and the effort P acts in the opposite direction, also

$$P = Q + R,$$

or

$$R = P - Q.$$

Thus, in order to lift a weight, the effort must be applied *upwards*, and the reaction of the fulcrum acts *downwards*; so that the lever presses *upwards* on the fulcrum.

Since CB > CA, the effort is greater than the weight; so that *the mechanical advantage is always less than unity.*

Examples of this class.—The treadle of a turning-lathe or scissors-grinding machine; the human arm.

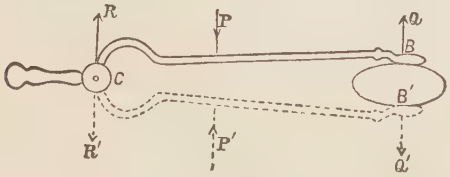


Fig. 97.

Double lever.—A pair of tongs (Fig. 97).

Levers of the third class are rarely used, for, since the mechanical advantage is less than unity, a greater effort is necessary to overcome the resistance than if no lever were used at all. This is sometimes expressed by saying that there is *mechanical disadvantage*. They are therefore useless for raising heavy weights. The use of a pair of tongs is to enable the effort to be applied at a more convenient position.

OBSERVATION.—In the solution of problems relating to levers, the student is recommended not to make use of the distinction between the three classes of lever, but to work each case out independently, either by taking moments or by writing down the conditions of equilibrium of the three forces acting on the lever.

107. **Conditions of equilibrium of a heavy lever.**—When the weight of a lever itself has to be taken into account, the condition of equilibrium may be found as in other cases by taking moments about the fulcrum.

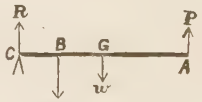


Fig. 98.

Let a lever, whose weight w acts at **G** and whose fulcrum is **C**, be used to overcome a resistance or raise a weight W applied at **B** by means of an effort applied at **A**. Then, if P denote this effort (considered positive when acting upwards), the equation of moments about **C** gives

$$P \,.\, \mathbf{CA} = W \,.\, \mathbf{CB} + w \,.\, \mathbf{CG} \quad \ldots\ldots\ldots \quad (1a) ;$$

and, if the lengths **CA, CB, CG** are considered positive or negative according to the direction in which they are drawn from **C**, this formula will be applicable to levers of any class.

The upward reaction (R) at the fulcrum will be given algebraically by $\qquad R + P = W + w.$

OBSERVATION.—These formulae hold good *whether the lever be horizontal or inclined to the horizon*, provided that the forces on the lever are parallel, and that their points of application lie in a straight line through the fulcrum.

108. The wheel and axle are two cylindrical rollers joined together with a common axis terminating in two pivots about which they can turn freely. The larger roller is called the **wheel,** and the smaller the **axle.** Both the wheel and the axle have ropes coiled round them in opposite directions. The rope on the axle supports the weight, and the effort is applied by pulling the rope

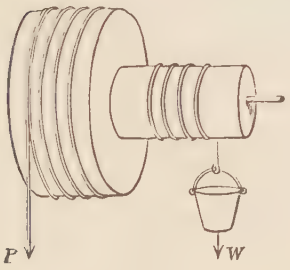


Fig. 99.

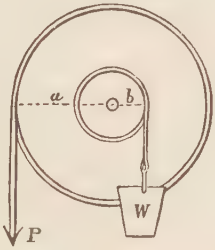


Fig. 100.

attached to the wheel. As the rope round the wheel unwinds, that round the axle winds up and raises the weight. Fig. 100 shows an end view of the arrangement.

Mechanical advantage of the wheel and axle.—The condition of equilibrium is the same as if the strings were really in a vertical plane perpendicular to the common axis as they appear in Fig. 100, and therefore the moments of the effort P and weight W about the axis are equal and opposite. Here, if a denotes the radius of the wheel, and b that of the axle, then a, b are the arms on which P and W act, and therefore

$$Pa = Wb\,;$$

i.e. effort × rad. of wheel = weight × rad. of axle.

$$\therefore \quad \text{mechanical advantage} = \frac{W}{P} = \frac{a}{b} = \frac{\text{rad. of wheel}}{\text{rad. of axle}} \qquad \ldots\ldots\ldots\ldots (2).$$

By making the wheel larger and the axle smaller, the mechanical advantage will be increased.

The wheel and axle is thus really equivalent to a lever whose arms are the radii of the wheel and the axle. But the lever can only be used for raising weights through short distances; the wheel and axle will lift them to any desired height.

Instead of the rope being coiled round the wheel, an endless rope may be used, passing round a groove cut in the rim of the wheel, as in a common roller-blind, provided proper precautions are taken to prevent the rope from slipping round the groove.

109. The windlass (Fig. 101) is a modification of the wheel and axle, the only difference being that the effort is applied by turning a handle **AH** at the end of an arm **CA**.

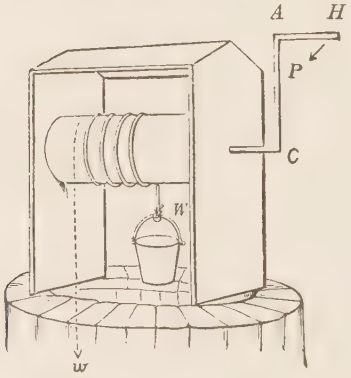


Fig. 101.

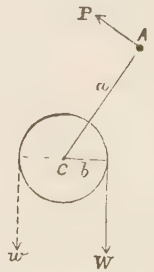


Fig. 102.

It is commonly used for raising buckets of water from a well, or earth from a shaft. An improved form has two buckets so arranged that the empty one goes down as the full one comes up.

Mechanical advantage.—If a is the length of the arm **CA**, the equation of moments gives, as before,

$$Pa = Wb;$$

and $\therefore$ mech. advantage $= \frac{a}{b} = \frac{\text{length of arm}}{\text{radius of axle}}$,

the length of the arm taking the place of the radius of the wheel.

If there are two buckets, and the total ascending and descending weights are W and w, we shall have, by taking moments,

$$Pa = Wb - wb = (W - w)b.$$

Example.—The axle of a windlass is 8 in. in diameter, and carries two buckets of equal weight on opposite sides. To find the force which must be applied to a handle, whose arm is 2 ft., to raise 3 gallons of water, a gallon weighing 10 lb.

Let P be the force, w the weight of each bucket. Then, since the radius of the axle is 4 in., the arm of the handle 24 in., and the total weights on the two sides w and $w + 30$ lb., we have, by moments,

$$P \times 24 + w \times 4 = (w + 30) \times 4;$$

whence $$P = 5 \text{ lb.}$$

[Notice that the weights of the two buckets balance each other.]

110. The capstan (Fig. 103) used on board ship is exactly similar in principle, but the barrel turns on a vertical axis

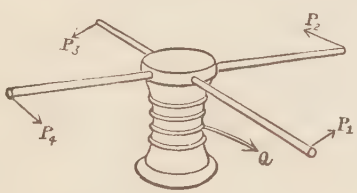


Fig. 103.

and is worked by one or more men walking round and pushing a number of horizontal projecting arms (called handspikes). Here the moment of the pull of the rope is equal to the *sum* of the moments of the forces exerted by the men.

Example.—The barrel of a capstan is 3 ft. in diameter, and is worked by four men exerting forces of 45, 52, 63, and 64 lb. on arms each $7\frac{1}{2}$ ft. long. The rope passing round the barrel is fastened to a pier. To find the force drawing the ship towards the pier.

Let Q lb. be the required force exerted by the rope. Then, since the radius of the barrel is $1\frac{1}{2}$ ft., we have, by taking moments,

$$Q \times 1\tfrac{1}{2} = (45 + 52 + 63 + 64) \times 7\tfrac{1}{2};$$

$$\therefore \quad Q = 224 \times 5 \text{ lb.} = 1120 \text{ lb.} = \tfrac{1}{2} \text{ ton.}$$

111. **The principle of work for machines.**—In this book we are usually assuming that all our machines work *without friction*, and that their parts have no weight. With such machines the following principle (called the Principle of Work) is universally true :—

The work done by the machine against the resistance is equal to the work done on the machine by the effort.

For example, if a workman using a frictionless windlass could raise a weight of 40 lb. by exerting a force of 5 lb., it would be found that in order to raise the weight 1 foot he would have to move the handle of the windlass through 8 feet. Thus the work done by the workman on the machine would be 5 lb. × 8 feet, or 40 ft.-lb. ; and the work done by the machine on the weight would be 40 lb. × 1 foot, which is also 40 ft.-lb.

This principle can easily be deduced from the principle of the "Conservation of Energy," for by this theory a body which does m ft.-lb. of work uses up m ft.-lb. of its energy, while a body on which work is done to the extent of n ft.-lb. obtains an increase of n ft.-lb. in its energy.

It follows that if a machine does m ft.-lb. of work and at the same time has work done on it to the extent of n ft.-lb., it will gain $(n - m)$ ft.-lb. of energy if $n > m$, or will lose $(m - n)$ ft.-lb. of energy if $m > n$.

It follows also that if the energy of the machine is being increased, the work which is done by the effort on the machine must be greater than the work which the machine is doing against the resistance ; and conversely. Now if there is friction between the parts of the machine, heat energy is being produced in it ; and hence more work is being done on the machine than the machine is doing against the resistance. Again, if the parts of the machine have weight, any parts which are being raised will be gaining potential energy, and any parts which are being lowered will be losing potential energy ; so that the net result may be either a gain or loss of energy, and a heavy machine may conceivably be doing either less or more work against the resistance than is being done on it by the effort.

It is now evident that if the machine works without friction, and if its parts are without weight (and if it contains no source of energy such as a compressed spring or a charged electric accumulator) the working of the machine produces neither gain nor loss of energy in the machine itself, and consequently the work which the machine does against the resistance is exactly equal to the work which is done on the machine by the effort.

The Principle of Work is thus an immediate corollary of the Principle of Conservation of Energy, and indeed it is sometimes called by the latter name.

111*a*. Let us consider again the case of the windlass quoted in the last section. By using an effort of 5 lb. the workman raises a weight of 40 lb.,—a considerable "gain in power." But to raise the weight through 1 foot he has to move the handle through 8 feet,—a corresponding "loss in speed." In the same way it follows from the principle of work that in any machine a gain in power is accompanied by a loss in speed. In more accurate language, mechanical advantage is always obtained at the expense of a proportionate disadvantage in diminished speed.

Conversely a gain in speed is accompanied by "mechanical disadvantage." For example, in the bicycle a rapid motion is produced by a comparatively slow movement of the feet on the pedals; but the force exerted by the foot is much greater than the actual resistance to the motion of the machine. Hence the difficulty experienced in riding up a slope or against a wind.

112. **Principle of Work for the wheel and axle.**—Let a, b be the radii of the wheel and axle, and let them be rotated through one complete turn. Then a length of rope equal to the circumference of the wheel, or $2\pi a$, uncoils from off the wheel, and a length equal to the circumference of the axle, or $2\pi b$, coils round the axle.

Hence, distance *fallen* by P (the effort)

$$= \text{circumference of wheel} = 2\pi a,$$

and distance *risen* by W (the weight)

$$= \text{circumference of axle} = 2\pi b\ ;$$

$\therefore$ work done by $P = P \times 2\pi a$,

and work done against $W = W \times 2\pi b$.

(i.) *If we assume the equation of moments*

$$P \times a = W \times b,$$

then $$P \times 2\pi a = W \times 2\pi b,$$

or work done by P = work done against W,

verifying the truth of the Principle of Work for the wheel and axle.

(ii.) Conversely, *if we assume the Principle of Work to be true,* then $P \times a = W \times b$,

verifying the relation between the effort and resistance, which is otherwise obtainable from the equation of moments.

113. To find the mechanical advantage of any machine from the Principle of Work.

We shall now show that the mechanical advantage or the condition of equilibrium of a machine working without friction can very easily be found by means of the Principle of Work when they *cannot* easily be found by other methods.

For let the machine be set in motion. Then it is only necessary to compare the distances through which the points of application of the effort and resistance move; their ratio is the required mechanical advantage.

For if P, Q denote the effort and resistance, x, y the distances moved by their points of application, then the relation work done by P = work done against Q

gives $$P \times x = Q \times y.$$

$\therefore$ mechanical advantage, $\dfrac{Q}{P} = \dfrac{x}{y}$

$$= \frac{\text{distance moved by point of application of } P}{\text{distance moved by point of application of } Q}$$

Example.—If, by moving a handle through 1 ft., a weight of 1 cwt. is raised through 1 in., to find the force that must be applied to the handle.

Let P be the force. Since the works of P and the resistance are equal and opposite,

$$\therefore \quad P \times 1 = 112 \times \tfrac{1}{12}.$$

Hence force required to raise 112 lb. $= 112/12 = 9\frac{1}{3}$ lb. weight.

114. Toothed or cog-wheels.—By the use of toothed wheels, the mechanical advantage of a windlass or other similar machine can be increased to any desired extent. By counting the number of teeth or cogs on two wheels which work into one another, we may find the number of turns and fractions of a turn made by one wheel for each turn of the other, and in this way determine the relation between the effort and weight.

Example.—The handle of a windlass is 1 ft. long, and is connected to a cog-wheel with 6 teeth, which works another cog-wheel with 84 teeth, connected to an axle 9 in. in diameter on which is coiled the rope supporting the weight. To find the force which must be applied to the handle to lift 5 cwt.

Since the cog-wheels contain 6 and 84 teeth, respectively,

$\therefore$ one turn of the axle corresponds to $\frac{84}{6}$ or 14 turns of the handle.

In one turn of the handle the extremity describes a circle of circumference 2π ft., and in one turn of the axle a length $\frac{3}{4}\pi$ ft. of rope coils up, drawing up the weight.

Therefore, by the Principle of Work, if P is the required force in lb. weight,

$$P \times 14 \times 2\pi = 560 \times \tfrac{3}{4}\pi.$$

$$\therefore \quad P = \frac{560 \times 3}{14 \times 2 \times 4} = 15 \text{ lb. weight.}$$

Illustrative Examples.

Examples (1).—A slab of stone weighing 1 ton, whose weight acts at its centre, is to be tilted up by a crowbar 3 ft. long resting against a log of wood in front of it. To find where the log must be placed in order that a force of 1 cwt. may suffice to raise the slab.

Let **ACB** be the crowbar, **BGD** the slab resting on the ground at **D**, **G** its centre of gravity. Let P, Q, R be the forces acting on the crowbar at **A**, **B**, **C**.

Then the weight of the stone, 1 ton (= W, say), acts at **G**, and is lifted about **D** by the force Q acting at **B**.

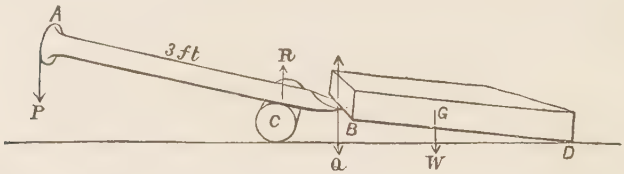


Fig. 103*a*.

Taking moments about the point of contact with the ground, it is clear that, since **G** is the centre of the slab **BD**;

$$\therefore \quad Q = \tfrac{1}{2}W = \tfrac{1}{2} \text{ ton} = 10 \text{ cwt.}$$

Taking moments about **C** for the crowbar,

$$P \times \mathbf{AC} = Q \times \mathbf{CB}.$$

Or, since $P = 1$ cwt., $\quad \mathbf{AC} = \dfrac{Q}{P}\mathbf{CB} = \dfrac{10}{1}\mathbf{CB} - 10\mathbf{CB}$;

$$\therefore \quad \mathbf{AB} = \mathbf{AC} + \mathbf{CB} = 10\mathbf{CB} + \mathbf{CB} = 11\mathbf{CB};$$

$$\therefore \quad \mathbf{CB} = \tfrac{1}{11}\mathbf{AB} = \tfrac{1}{11} \text{ of } 3 \text{ ft.} = 3\tfrac{3}{11} \text{ in.}$$

Hence the crowbar must rest on the log at a point $3\frac{3}{11}$ in. from its extremity.

(2) The driving wheel of a bicycle is 90 in. in circumference, the cranks of the pedals are $4\frac{1}{2}$ in. long, and the driving wheel makes 20 turns for every 9 turns of the crank axle. If the force resisting the motion of the machine is $\frac{1}{2}$ lb., to find the average force which the rider exerts on the pedals, supposing him to press on them vertically downwards.

In each turn of the driving wheel the machine moves forward 90 in.

But the turn of the cranks produces $\frac{20}{9}$ turns of the driving wheel.

Therefore in one turn of the *cranks* the machine moves forward 200 in. $= \frac{50}{3}$ feet, and the work done against the resistance is $\frac{50}{3} \times \frac{1}{2}$ or $\frac{25}{3}$ ft.-lb. weight.

But in one turn of the cranks each of the two pedals is lowered in turn through a vertical distance = twice the length of the crank = 9 in. $= \frac{3}{4}$ ft.

Hence, if P lb. denote the average force which the rider exerts on the pedals, the Principle of Work gives

$$P \times 2 \times \tfrac{3}{4} = \tfrac{25}{3}.$$

$$\therefore \quad P = \tfrac{50}{9} = 5\tfrac{5}{9} \text{ lb. weight.}$$

(3) The pedals of a bicycle describe a circle of radius 6 inches ; and the machine is " geared " to 72 inches. Find the pressure of the foot on the pedal when riding at a uniform rate up a slope of 1 in 20 ; given that the road resistance is 1 lb., the air resistance is $\frac{1}{2}$ lb., the weight of the machine is 40 lb., and of the rider 10 stone. Assume that the rider exerts a uniform pressure on each pedal *in the direction of its motion* during exactly one-half of its revolution.

Consider the work done on the machine by the feet, and by the machine against the various resistances, during one revolution of the pedal.

The machine is geared to 72 inches ; this means that the distance moved by the machine during one revolution of the pedals is equal to the circumference of a wheel of diameter 72 inches, that is to 6π feet. Hence the work done against the road and air resistances is $6\pi \times 1\frac{1}{2}$ or 9π ft.-lb.

Again the machine and rider (total mass 180 lb.), are raised a vertical height of $\frac{1}{20} \times 6\pi$ feet. Hence the work done against gravity is $180 \times \frac{1}{20} \times 6\pi$ or 54π ft.lb.

Thus the total work done against resistances amounts to 63π ft.-lb.

In one revolution each pedal travels π feet, and is acted upon by a pressure x lb. during one-half of this distance. Thus the total work done on the two pedals is $2 \times x \times \frac{1}{2}\pi$ or πx ft.-lb.

Hence by the principle of work $\pi x = 63\pi$, whence $x = 63$ lb.

(The result is large owing to the steep gradient.)

The last two examples should be carefully compared, especially with regard to the different assumptions as to the manner in which the effort is applied, and the effect on the calculations. Probably the first assumption is more correct.

(4) The arms of a light bent lever measure respectively 8 and 24 inches ; and they include an angle of 75°. If a mass of $9\sqrt{2}$ lb. is attached to the end of the shorter arm, and another of 3 lb. to the end of the longer arm, find the position of equilibrium.

Let θ be the angle **AFH** between the shorter arm and the horizontal.

Then $\mathbf{KFB} = 105° - \theta$.

Also if **AH**, **BK** are the verticals through **A** and **B**,

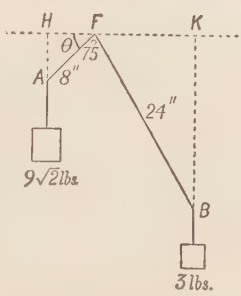


Fig. 103*b*.

$$\mathbf{FH} = \mathbf{FA} \cos\theta = 8 \cos\theta,$$

and $$\mathbf{FK} = \mathbf{FB} \cos(105° - \theta) = 24 \cos(105° - \theta),$$

Taking moments about **F** we obtain $9\sqrt{2} \times \mathbf{FH} = 3 \times \mathbf{FK}$,

i.e. $$9\sqrt{2} \times 8 \cos\theta = 3 \times 24 \cos(105° - \theta),$$

or $$\sqrt{2} \cdot \cos\theta = \cos(105° - \theta),$$

whence $$\sqrt{2} \cdot \cos\theta = \cos 105° \cdot \cos\theta + \sin 105° \cdot \sin\theta.$$

Dividing by $\cos\theta$; $$\sqrt{2} = \cos 105° + \sin 105° \cdot \tan\theta,$$

i.e. $$\tan\theta = \frac{\sqrt{2} - \cos 105°}{\sin 105°}.$$

Now $\cos 105° = \cos(45° + 60°) = \frac{1 - \sqrt{3}}{2\sqrt{2}}$; $\sin 105° = \frac{1 + \sqrt{3}}{2\sqrt{2}}$.

Substituting these values we obtain

$$\tan\theta = \sqrt{3}, \quad \text{whence} \quad \theta = 60°.$$

(5) In a wheel and axle, the radius of the wheel is a and of the axle is b, the weight is W, and the rope at which the effort is applied slopes at an angle θ to the vertical. Determine the effort, and the magnitude and direction of the thrust on the bearings of the machine.

We assume as usual that the machine has no weight and is in equilibrium.

Taking moments about **O**, we obtain $P \cdot a = W \cdot b$; whence

$$P = \frac{b}{a} W \ldots\ldots\ldots \text{(i)}.$$

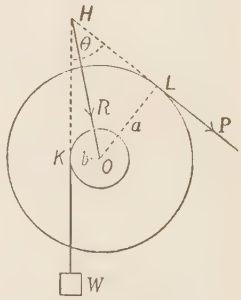


Fig. 103c.

Now the thrust R on the bearings is the resultant of P and W (for these two forces produce this thrust). Since the angle between P and W is θ, we have by § 31

$$R = \sqrt{(P^2 + W^2 + 2PW\cos\theta)},$$

or, substituting from (i),

$$R = \frac{W}{a}\sqrt{(a^2 + b^2 + 2ab\cos\theta)} \quad \text{(ii)}.$$

Again, by § 31, if ϕ be the inclination of R to W we have

$$\tan\phi = \frac{P\sin\theta}{W + P\cos\theta},$$

or, substituting from (i),

$$\tan\phi = \frac{b\sin\theta}{a + b\cos\theta} \quad \ldots\ldots\ldots\ldots \quad \text{(iii)}.$$

Note that the line of action of R must pass through **O**; for the moment of R about **O** is equal to the algebraic sum of the moments of P and W about **O** (§ 69), and is therefore zero.

NOTE.—The student is at liberty to apply the Principle of Work or Principle of Conservation of Energy to any problem whatever in Mechanics, provided its use is not precluded by the conditions of the question (as, for example, where it is required to *verify* the principle, when its truth must not be *assumed*).

114*a* **Efficiency.**—We have seen that in frictionless machines the work done by the machine is equal to the work done on the machine. In any actual machine, however, there is friction to be overcome, and accordingly the work done by the machine is always less than that done on the machine. Remember the following definition :—

$$\text{Efficiency of a machine} = \frac{\text{work done by machine}}{\text{work done on machine}}.$$

It follows from this definition that the efficiency of a machine is always less than unity : for a frictionless machine it would be unity.

Example.—In a system of pulleys a man using a force P lb. raises a weight W lb. through h ft. by pulling his end of the rope through s ft. Find the efficiency of the machine.

$$\text{Efficiency} = \frac{\text{work done on the weight}}{\text{work done by the man}} = \frac{Wh}{Ps}.$$

SUMMARY OF RESULTS.

Mechanical advantage of any machine is defined as fraction

$$\frac{\text{resistance to be overcome}}{\text{effort applied to move machine}} = \frac{Q}{P} \text{ or } \frac{W}{P}. \quad (\S\ 99.)$$

Let this be denoted by M, while Q or W and P have their usual meanings.

In any *lever* the equation of moments about fulcrum gives

$$M \text{ or } \frac{Q}{P} = \frac{\text{distance of } P \text{ from fulcrum}}{\text{distance of } Q \text{ from fulcrum}} \ldots (1). \quad (\S\S\ 101, 102.)$$

For a *straight lever* **ABC**, whose fulcrum is **C**, the conditions of equilibrium are

$$\frac{P}{\mathbf{BC}} = \frac{Q}{\mathbf{CA}} = \frac{R}{\mathbf{AB}}, \quad (\S\ 103.)$$

where R is the reaction of the fulcrum.

In a lever of the

first class, *fulcrum* is in middle, $M > = \text{or} < 1$, (§ 104.)

second ,, *resistance* ,, ,, M always > 1, (§ 105.)

third ,, *effort* ,, ,, M always < 1. (§ 106.)

In a *wheel and axle* or *windlass*, $Pa = Qb$, or

$$M = \frac{a}{b} = \frac{\text{rad. of wheel}}{\text{rad. of axle}} \text{ or } \frac{\text{arm of handle}}{\text{rad. of axle}} \ldots\ldots \quad (2).$$

(§§ 108, 109.)

The Principle of Work for any frictionless machine gives

work of effort = work against resistance ;

whence, if x, y are distances traversed by P, Q,

$$Px = Qy, \qquad M = \frac{x}{y}. \qquad (\S\ 113.)$$

For any machine working with friction

$$\textit{efficiency} = \frac{\textit{work done by machine}}{\textit{work done on machine}} \qquad (\S\ 114a.)$$

EXAMPLES VIII.

(*Selected Problems* :—2, 3, 6, 7, 9, 12, 14, 16, 18, 19, 21, 24, 26, 27, 28.)

1. The arms of a lever are 4 in. and 11 in. in length, and a weight of 41¼ lb. is attached to the shorter arm. Find the power.

2. A lever, 6 ft. long, having the fulcrum at one end and the applied force at the other end, is used to sustain a weight of 1 cwt. at a point 2 ft. from the fulcrum. Find the direction and magnitude of the pressure on the fulcrum.

3. Two weights P and Q balance on a weightless lever, the fulcrum being 1¼ in. from the middle point of the lever. If each weight is increased by 1 lb., the fulcrum must be moved ¼ in. in order that there may be equilibrium. Find the force of pressure on the fulcrum in each case.

4. Explain carefully why a man stands on the bottom rung of a ladder, and holds on to another rung as low down as he can, when another man is lifting the ladder.

5. Two men carry a load of 1 cwt. suspended from a horizontal pole 12 ft. long, whose weight is 20 lb., and whose ends rest on their shoulders. Find the point at which the load must be suspended in order that one of the men may bear 94 lb. of the whole weight.

6. The drum of a windlass is 4 in. in diameter, and the power is applied to the handle 20 in. from the axis. Find the force necessary to sustain the weight of 100 lb., and the work done in turning the handle ten times.

7. Why cannot a man, sitting in a basket, lift himself and the basket off the ground by pulling at the handles of the basket?

8. What power will balance a weight of 6 cwt. by means of a wheel and axle whose circumferences are, respectively, 5 ft. 4 in. and 9 in.?

9. A wheel and axle is used to raise a bucket weighing 30 lb. from a well. The radius of the wheel is 20 in., and while it makes 7 revolutions the bucket rises 11 ft. What is the smallest force that will raise the bucket?

10. Two forces P and Q balance on a lever acting on the same side of the fulcrum. If P be increased by 1 lb., equilibrium may be maintained by moving P's point of application 2 in.; or if Q be increased by 1 lb., by moving Q's point of application $\frac{3}{4}$ in.; or if both P and Q be increased by 1 lb., by moving both points of application $1\frac{1}{5}$ in. Find P and Q.

11. In any lever, find the mechanical advantage when the effort is applied at (i.) the point of application of the resistance, (ii.) the fulcrum. Applying this to the case of an oar in a boat, state what happens when the oarsman pulls hold of it *at* the rowlock, and hence show that the oar acts as a lever whose fulcrum is at the rowlock.

12. **AB** is a weightless lever acted on at **A** and **B** by two equal forces P and Q, whose directions contain an angle 60°, P acting at right angles to **AB**. Find where the fulcrum must be situated so that P and Q may be in equilibrium, and find the force of pressure they exert on the fulcrum.

13. **AB** is a weightless rod turning freely round its middle point **C**; **D** is a point vertically under **C**, such that **CD** equals half the length of the rod; the points **B** and **D** are connected by a thread of the same length as **CD** or **CB**. If a weight W is hung from **A**, what is the tension of the thread **BD**, and what is the magnitude and direction of the pressure on the point **C**?

14. Two weights of 4 lb. and 8 lb. balance when suspended from the ends of a straight lever with the fulcrum 1 ft. from the larger weight. When P lb. are added to each weight, the fulcrum has to be shifted a distance of 2 in. Find the value of P and the length of the lever.

15. Draw to scale a wheel and axle by which a man, sitting in a loop at the end of a rope wound round the axle, can haul himself up by pulling at a rope round the wheel with a force only one-fifth of his weight. What weight is sustained by the pivots?

16. State the principle of the Conservation of Energy, and show how it holds good in the case of a lever of the third kind.

17. In a weightless straight lever of the first order, show that in the case of equilibrium the power, the weight, and the reaction of the fulcrum form two unlike couples of equal moments.

18. A straight horizontal lever has for fulcrum a hinge at one end **A**, and at a point **B** is hung the weight W. If the strain on the hinge must not exceed $\frac{1}{3}$ W either upwards or downwards, prove that the power must act somewhere within a space equal to $\frac{3}{4}$ **AB**.

19. A straight lever, whose arms are a and b feet respectively, is in equilibrium under forces P and Q at its extremities, whose lines of action make, with the lever, angles α and β respectively. Find the ratio of P to Q, and the pressure on the fulcrum.

20. **ACB** is a lever, made from a uniform rod of weight W bent at the fulcrum **C**. The lengths of the arms are a and b. If no weights are attached to the lever it rests with **AC** horizontal. What weight must be attached at **A**, in order that it may rest with **BC** horizontal?

21. **AB** is a lever whose fulcrum is at its middle point **F**. At **B** is attached a weight of $\sqrt{3}$ lb. A string attached to **A** passes over a smooth peg **C**, and supports a weight of 1 lb. Given that **AC** is horizontal and equal to **FA**, find the inclination of the lever to the horizontal in the position of equilibrium.

22. A light lever **BD** is hinged at **B**; its upper end **D** supports a weight W. It is kept in equilibrium by a light rod **AC**, which is fixed at **C**, and fastened to the lever at its middle point **A**. If **BA** and **CA** are inclined at angles α and β to the vertical, find the thrust in the rod **AC**.

23. A uniform rod, 12 in. long, and weighing 10 lb., is hinged at one end to a vertical wall, and supported in a horizontal position by a string which is fastened to the rod at a distance 5 in. from the hinge, and to the wall at a height 12 in. above the hinge. If a weight of 20 lb. is suspended from the end of the rod, find the tension of the string, and the direction of the reaction at the hinge.

24. A light lever 26 in. long is hinged at one end to a wall and carries a weight at the other end. It is supported by a string 20 in. long, one end of which is fastened to the middle of the lever, and the other to a point on the wall 11 in. above the hinge. Find the direction of the reaction at the hinge.

25. **CD** is a uniform rod, of weight W and length l. It is hinged at its upper end **C** and its lower end is fastened by a string of length b to a light ring which slides on a smooth vertical rod **AB**. **AC** is horizontal and of length a. Given $a > b$, find the tension of the string.

26. A man, in sculling, pulls with a force of 40 lb. wt. on the handle of each scull; two-thirds of each scull projects beyond the rowlock. Assuming that the forces on each scull are practically in equilibrium, determine (*a*) the stress at each rowlock, (*b*) the stress at each blade, (*c*) the total force urging the boat and man forwards. Why is this not equal to the sum of the forces at the rowlocks?

27. A cyclist is travelling at a uniform rate of 15 miles per hour, and the pedals are making 7 revolutions in 4 seconds. If the pedals describe a circle of radius 6 in., and if there is on the average a pressure on the one pedal or the other of 2 lb. wt., determine the total external resistance to the motion, neglecting internal resistances and using the principle of work. (Assume the force on the pedal always perpendicular to the crank.)

28. Determine the mechanical advantage of a bicycle from the following data :—diameter of driving wheel = 30 in. ; number of cogs on the cog-wheel attached to it = 15 ; radius of the circle described by the pedals = 6 in. ; number of cogs on the cog-wheel attached to the pedals = 40.

CHAPTER IX.

MACHINES—THE PULLEY AND SCREW.

115. The pulley, or pully,* is a wheel with a *groove* cut round its rim so that it can carry a string or rope or chain passing round it.† It turns on an axis or axle, which is fixed in a framework called a *block* or *sheave*, and this block is either **fixed**, or is attached to a string and is then **moveable**.

Fig. 104.

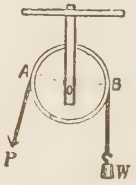


Fig. 105.

116. In the fixed pulley the weight is attached to one end of the string passing round the groove, and the effort is applied by pulling the other end. Since the wheel is only supported by the axle, the moments of the effort P and weight W about the centre **O** are equal and opposite; that is (Fig. 105),

$$P \times \mathbf{OA} = W \times \mathbf{OB},$$

* The word may either be spelt *pulley*, plural *pulleys*, or *pully*, plural *pullies*.

† For the sake of uniformity we shall speak of a *string*. The weight of this string will be assumed too small to require to be taken into account.

or $P \times$ radius of pulley $= W \times$ radius of pulley.

$$\therefore \quad P = W,$$

$$\therefore \text{ mechanical advantage } \frac{W}{P} = 1 \ldots\ldots\ldots \quad (1).$$

Cor. Since $P = W$, the tension is the same in both parts of the string.

Thus exactly the same force must be applied to lift a given weight as if the weight were lifted without the pulley. The only difference is that the force can be applied in a different direction ; hence the usefulness of the fixed pulley lies in this convenience only.

Thus, in raising building materials to the top of a house, it is far easier for a man at the bottom to pull down a rope passing over a pulley at the top than it would be for a man at the top to hoist them up with a rope.

117. In the single moveable pulley, the weight is attached to the block, and the effort is applied to one end of the string which passes round the pulley, the other end being fixed up.

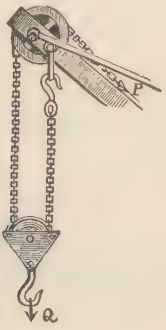


Fig. 106.

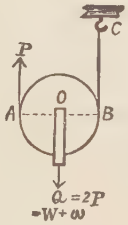


Fig. 107.

In this arrangement, if the strings are parallel, the mechanical advantage is 2.

For let P be the effort, Q the total weight to be raised (*including the weight W of the pulley itself*). Then the tension in each part of the string is equal to P (§ 116); also the tensions at **A, B** (Fig. 107) both help to support the weight Q.

$$\therefore \quad Q = 2P,$$

and **mechanical advantage** $Q \div P = 2$ ..... (2).

118. If w is the weight of the pulley itself, W the weight of the attached load, $Q = W + w$; and, therefore,

$$W + w = 2P \text{ .. (2a).}$$

The single moveable pulley is much used on cranes (Fig. 106).

119. The single-string system of pulleys.*—A greater mechanical advantage may be obtained with a number of pulleys. Several such "**systems of pulleys**" are generally described, but the most practically useful system is that in which the pulleys are arranged in two blocks, one fixed and the other attached to the weight (Fig. 108). The same string passes round all the pulleys;

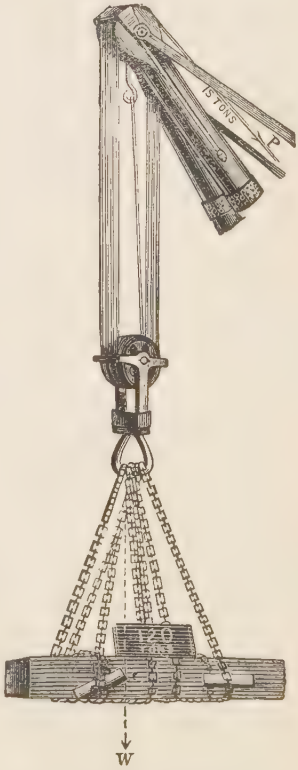


Fig. 108.

* The single-string system is often called the *second system* of pulleys, and the separate-string system of § 121 is then called the *first system*.

it passes alternately over a fixed and under a moveable pulley, and is finally attached to one of the blocks.

In practice the pulleys are arranged as in Fig. 108, but it is generally easier to draw the diagram as in Fig. 109, with all the pulleys in the same plane.

Mechanical advantage.—If the string be pulled with a force P, the tension throughout is P. Hence, if n be the number of parts of the string supporting the lower block, Q the weight to be raised (including that of the lower block and pulleys), the n upward forces P support Q acting downwards; hence, supposing the parts of the string vertical,

$$Q = nP.$$

$$\therefore \text{ mechanical advantage } \frac{Q}{P} = n \quad \ldots\ldots (3).$$

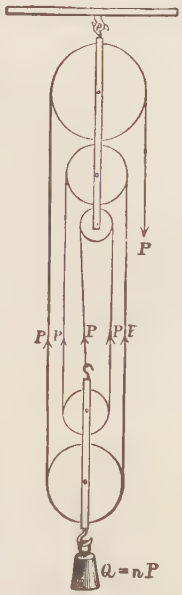


Fig. 109.

120. If w is the weight of the lower block, W that of the attached load, $Q = W + w$; and, therefore,

$$W + w = nP \quad \ldots\ldots (3a)$$

In order that the effort may be applied *downwards*, the free end of the string must hang from a fixed pulley, and this is almost invariably done for convenience in working the system. In such cases the number n is also the total number of pulleys in the two blocks.

Thus Fig. 108 represents a system with altogether 8 pulleys, in which the mechanical advantage is therefore 8. In Fig. 109, there are altogether 5 pulleys, and the mechanical advantage is 5.

Example.—To find the least number of pulleys in a moveable block weighing 10 lb., in order that a weight of 120 lb. may be lifted by a downward force not exceeding 28 lb., and to find this force.

Let n be the total number of portions of the string supporting the lower block, P the required effort. Then the pulls P in the strings have to support both the attached weight of 120 lb. and the block weighing 10 lb.; therefore

$$nP = 120 + 10 = 130 \text{ lb.}$$

But P is *not more* than 28 lb.; therefore n must be greater than $4\frac{9}{14}$; that is, $n = 5$.

Hence five parts of the string must support the lower block. Therefore that block must contain two pulleys, and must have the end of the string attached to it as well (Fig. 109). Also, putting $n = 5$, we have

$$5P = 130 \text{ lb.}; \quad \therefore \quad \text{required force } P = 26 \text{ lb.}$$

121. **The separate-string system of pulleys*** consists of a number of single moveable pulleys like that described in § 117, so arranged that the string hanging from one pulley passes round the pulley next below, the other ends of the strings being attached to a fixed beam or other support (such as the mast of a ship), considerably above the highest points to which weights have to be raised (Fig. 110).

The mechanical advantage may be found as follows:—

In the single moveable pulley a force P applied to the string supports a force $2P$ applied to the block.

Now suppose the moveable pulley, instead of being attached to the weight, supports a string passing under a second moveable pulley. Then the mechanical advantage gained by the first pulley is evidently doubled by the second, the pull $2P$ in the second string supporting a weight $4P$ attached to the second pulley.

Next suppose the second pulley supports a string passing

* The so-called *first system.*

under a third pulley. This again doubles the mechanical advantage, and the system will now support a weight $8P$.

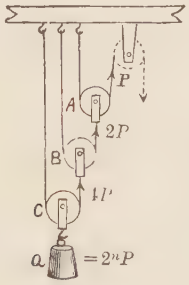


Fig. 110.

In this way each additional pulley doubles the mechanical advantage of the system. By using 1, 2, 3 pulleys, we get mechanical advantages 2, 4, 8.

Generally, let there be n pulleys, and let Q denote the weight attached to the last pulley. Then, if we leave out of account the weights of the pulleys themselves, we have

$$Q = 2^n P.$$

Therefore also $P = \frac{Q}{2^n}$

and mechanical advantage $= 2^n$ } ..... (4).

122. If the weights of the pulleys **A, B, C** are w_1, w_2, w_3 respectively, we must consider the equilibrium of each pulley separately. If T_1, T_2 be the tensions of the strings hanging from **A, B,** and W the load attached to **C,** we have, by § 118,

$$2P = T_1 + w_1, \quad 2T_1 = T_2 + w_2, \quad 2T_2 = W + w_3.$$

From equations such as these, it may be deduced that, for n pulleys,

$$2^n P = W + w_n + 2w_{n-1} + 4w_{n-2} + \ldots + 2^{n-1}w_1 \ldots\ldots (4a).$$

Examples.—(1) If there are (n =) 4 moveable pulleys, a force of (P =) 10 lb. will support a weight

$$(Q =)\ 2^4 P = 2^4 \times 10 \text{ lb.} = 160 \text{ lb.}$$

(2) If there are (n =) 3 moveable pulleys, the force required to support a weight (Q =) 64 lb. is

$$(P =)\ Q \div 2^n = 64 \div 2^3 = 64 \div 8 = 8 \text{ lb.}$$

(3) What load can be supported by a force of 10 lb. in a system of 3 moveable pulleys whose weights, beginning with the highest, are 1, 2, 3 lb., respectively?

Consider, first, the equilibrium of the highest pulley (**A**, Fig. 110). The forces on it are the two equal pulls of 10 lb. in the two parts of the string round it, acting upwards, and the weight of the pulley (1 lb.) and the tension T_1 of the string next below, acting downwards.

Hence $$2P = 1 + T_1 \ldots\ldots\ldots\ldots\ldots\ldots \quad (1).$$

Consider, now, the pulley **B**, acted upon by $2T_1$ upwards, and by T_2 and its own weight (2 lb.) downwards.

Hence $$2T_1 = 2 + T_2 \ldots\ldots\ldots\ldots\ldots\ldots \quad (2).$$

Similarly, for the pulley **C**, we get the equation of forces

$$2T_2 = 3 + W \ldots\ldots\ldots\ldots\ldots\ldots \quad (3).$$

Since P = 10, we find, from equation (1), that $T_1 = 20 - 1 = 19$; then, from equation (2), $T_2 = 2T_1 - 2 = 38 - 2 = 36$; and lastly, from equation (3), $W = 2T_2 - 3 = 72 - 3 = 69$ lb.

(4) What force is required to support a load of 13 lb. in a system of 4 moveable pulleys whose weights, commencing with the highest, are 3, 5, 7, 9 lb., respectively?

Here we are given the weight, and have to find the effort. Hence the process is the reverse of that of Ex. (3). We must begin with the lowest pulley. Adding its weight (9 lb.) to the attached weight (13 lb.), we have the total weight supported by the two pulls (T_3) in the parts of the string round the lowest pulley.

Hence $2T_3 = 22$ lb., $T_2 = 11$ lb.

Similarly, in the lowest pulley but one, we have 11 + 7 lb. supported by $2T_2$.

Hence $2T_2 = 18$ lb., $T_2 = 9$ lb.

In like manner, $2T_1 = 9 + 5$, $T_1 = 7$ lb.

$2P = 7 + 3$, $P = 5$ lb.

That is, the required force = 5 lb.

*123. **The inverted separate-string system of pulleys,*** in which the strings are all attached to the weight, is merely the system last described (*i.e.* the "first") turned upside down (the fixed pulley being omitted). The strings are all attached to the weight or rather to a rod, carrying the weight, and the uppermost pulley is fixed to some support. If Fig. 111 be turned upside down, it will present a similar appearance to Fig. 110.

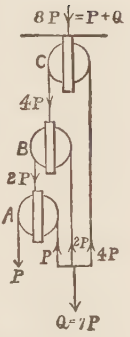


Fig. 111.

The mechanical advantage (when the weights of the individual pulleys are neglected) is easily deduced from this property. Let R be the pull which the system exerts on its support, Q the weight, P the effort, and let there be n pulleys. By inverting the system or otherwise, we see that the total forces supported by the several pulleys, commencing with the lowest, are $2P$, $4P$, $8P$, etc.; thus R in this present system corresponds to the weight in the last system, and therefore

$$R = 2^n P.$$

Now consider the equilibrium of the whole system, consisting of the weight and the pulleys. The forces acting on it are Q and P pulling downwards *and a reaction equal and opposite to R holding the system up.* Hence, since these forces keep the system in equilibrium,

$$R = P + Q.$$

$$\therefore \quad Q = R - P = 2^n P - P = (2^n - 1)P,$$

and **mechanical advantage** $= Q \div P = 2^n - 1 \ldots .$ (5).

Or thus : The pulls of the strings, beginning with the lowest, are P, $2P$, $4P$, $8P$, $..2^{n-1}P$, and, since these support the weight Q, we have $Q = P(1 + 2 + 4 + 8 + .. + 2^{n-1})$. Now $1 + 1 = 2$; therefore $1 + 1 + 2 - 2 + 2 = 4$; therefore $1 + 1 + 2 + 4 = 4 + 4 = 8$ and so on; therefore $1 + 1 + 2 + .. + 2^{n-1} = 2^{n-1} + 2^{n-1} = 2 \times 2^{n-1} = 2^n$ or $1 + 2 + 4 + .. + 2^{n-1} = 2^n - 1$, giving $Q = (2^n - 1)P$, as above.

* The so-called *third system.* This system is practically useless (see § 125).

Example.—To find the number of weightless pulleys, having given that a force of 5 lb. supports a weight of 75 lb., and all the strings being attached to the weight.

Here the pull on the beam supporting the upper pulley

$$= 75 \text{ lb.} + 5 \text{ lb.} = 80 \text{ lb.}$$

Now $80 = 5 \times (2 \times 2 \times 2 \times 2)$, and, since the total pull is doubled by each pulley, the number of pulleys must be 4.

124. *If the weights of the pulleys are taken into account,* we proceed as follows :—

Examples.—(1) To find the weight which can be lifted by a force of 10 lb. in a system of three pulleys, the weights of the lowest and next pulleys being 3 and 5 lb,* and the strings all being attached to the weight.

Here the pulls in the two parts of the lowest strings are each 10 lb. The next string above has to support these pulls, and also to support the weight of the lowest pulley, viz. 3 lb., and so on ; hence the process stands thus :

Pull of lowest string		= 10 lb.	= 10 lb.
Multiply by		2	
		20 ,,	
Add weight of lowest pulley		3 ,,	
∴ pull of second lowest string		= 23 ,,	= 23 ,,
Multiply by		2	
		46 ,,	
Add weight of second pulley		5 ,,	
∴ pull in last string		= 51 ,,	= 51 ,,

Required weight = sum of pulls of strings = 84 lb.

(2) To find the force required to support a weight of 112 lb. in the system of Ex. (1).

Let P be the force. Proceed, as in Ex. (1), thus :

Pull of lowest string	$= P$	$= P.$
Pull in second lowest string T_1	$= 2P + 3$	$= 2P + 3$
Pull in last string T_2	$= 2T_1 + 5$	$= 4P + 11$

$\therefore$ weight supported $= 7P + 14$

$\therefore \quad 7P + 14 = 112.$

$\therefore$ required force $P = 14$ lb.

* Since the upper pulley is *fixed*, its weight does not enter into the relation between P and W.

125. **Uses of the various systems of pulleys.**—The single-string system, with the pulleys arranged in two blocks, as in Fig. 108, is by far the most important system for practical use, because with it weights may be raised till the two blocks come together, *i.e.* nearly up to the level of the supporting beam or crane, or lowered till all the string is paid out.

The separate-string system gives a greater mechanical advantage with the same number of pulleys, for each additional pulley doubles the mechanical advantage instead of merely increasing it by unity. But it requires the strings to be attached to a point considerably above the highest point to which weights have to be raised, for when the top pulley is drawn up as high as it will go the weight will still be a considerable distance below the support. By attaching the strings (or rather ropes) high up on the mast of a ship, the system may be used, and *is* actually so used, for raising cargo from the hold up to the deck.

The inverted separate-string system, or " third system," is of no practical use whatever, for it is found that the strings are almost certain to get hopelessly entangled, even in the few working models that are constructed for the lecture room.

[This system seems to have been originally introduced into text-books and examination papers on account of its being a convenient subject for problems illustrating the summation of a geometrical progression.]

126. **Man raising himself with a system of pulleys.** - When a man, sitting in a loop or seat suspended by any arrangement of pulleys, pulls *himself* up, the rope which he pulls will support part of his weight, and only the remaining part of his weight will have to be supported by the system.

Generally, suppose that a man is pulling himself up by applying a force P to a rope passing round a system of pulleys or a wheel and axle, in which P would support a weight nP. Then, since the rope he pulls also helps lift him up with a force P, the total weight supported must

be $nP + P$ or $(n + 1)\,P$. Hence

$$\text{mechanical advantage} = n + 1 \quad \ldots\ldots\ldots (6);$$

instead of being n, as it would be for a man standing on the ground and raising a weight.

Examples.—(1) Consider a man of weight W raising himself by pulling a rope passing over a *fixed* pulley with a force P. The man's weight is really supported by the pulls P in *two* parts of the rope—the part where he pulls and the part supporting the loop. Hence the condition of equilibrium gives $W = 2P$;

$$\therefore \quad P = \tfrac{1}{2}W;$$

or the man pulls with a force of *half* his weight.

(2) If the man pulls a rope which passes over a fixed pulley and under a moveable pulley supporting the loop, there are three portions of the string supporting his weight, inclusive of the one that he is pulling. Hence the relation between the pulling force P and the weight is $W = 3P$, or $P = \frac{1}{3}W$; so that the man pulls with a force of *one-third* of his weight.

127. **Applications of the Principle of Work.**—We shall now apply the Principle of Work to find the relations between the effort and weight in the various systems of pulleys.

The single moveable pulley.—Let the pulley (weight w) and its attached weight W be raised through a height h. Then the portions of string on the two sides of the pulley will each be shortened by h; hence the end at which P is applied will rise through a distance $2h$. Since

sum of works done against W, w = work done by P:

$$\therefore \quad Wh + wh = P \times 2h,$$

or

$$W + w = 2P \quad \ldots\ldots\ldots\ldots (2a),$$

the required relation between W and P. This agrees with (2) on writing $Q = W + w$.

128. **The single-string system.**—Let there be n strings from which the lower block of pulleys hangs. If the effort P lifts a weight W, hanging from a moveable block of weight w, through a height h, each of the n portions of

string will have shortened by an amount h, and so P will have to pull the end of the string down through a distance nh. By the Principle of Work, therefore,

$$Wh + wh = nPh.$$

$$\therefore \quad W + w = nP \quad \ldots\ldots\ldots\ldots\ldots (3a),$$

the required relation between W and P, which agrees with (3) on writing $Q = W + w$.

*129. **The separate-string system.**—Let w_1 be the weight of the pulley supporting the weight W; let $w_2, w_3, \ldots$ be the weights of the pulleys next above. If W is drawn up through a height h, then the lowest pulley w_1 will rise through h, the next above, w_2, will rise through $2h$, the next, w_3, through $4h$, and so on; and if there are n moveable pulleys the force P will move its point of application through a distance $2^n h$.

Therefore, by the Principle of Work,

$$P \times 2^n h = Wh + w_1 h + 2w_2 h + 4w_3 h + \ldots + 2^{n-1} w_n h,$$

or
$$2^n P = W + w_1 + 2w_2 + 4w_3 + \ldots + 2^{n-1} w_n;$$

or again
$$P = \frac{W}{2^n} + \frac{w_1}{2^n} + \frac{w_2}{2^{n-1}} + \frac{w_3}{2^{n-2}} + \ldots + \frac{w_n}{2} \quad \ldots\ldots (4b).$$

[Note that w_1 stands for the weight of the *lowest* pulley in (4*b*) and the *highest* in § 122 (4*a*).]

*130. **The inverted separate-string system.**—In the system of § 129, when the weight rises through a height h, the various pulleys rise through distances $h, 2h, 4h, \ldots 2^{n-1}h$, and the point of application of P through $2h$. Hence *their distances from the weight* W increase by amounts $0, (2-1)h, (4-1)h \ldots (2^{n-1}-1)h, (2^n-1)h$.

Now turn the system upside down; fix the pulley that previously supported the weight, and attach the weight to the beam that was previously fixed. We now get the "third" system, and we see that when the weight rises through h the pulleys, beginning with the top, descend through distances $0, (2-1)h, (4-1)h \ldots (2^{n-1}-1)h$, and the point of application of P through $(2-1)h$. Hence if $w_1, w_2, \ldots$ are the weights of the pulleys, the Principle of Work gives

$$Wh = w_1 + 0h + w_2 \times h + w_3 \times 3h + w_4 \times 7h + \ldots + w_n \times (2^{n-1} - 1)h + P \times (2^n - 1)h,$$

or
$$W = (2^n - 1)P + w_2 + 3w_3 + 7w_4 + \ldots (2^{n-1} - 1)w_n \quad (5a).$$

the required relation connecting P and W.

131. The screw.—Every one is familiar with a **screw.** It consists essentially of a cylindrical bolt **OM,** whose surface carries a *thread* or has a *groove* cut in it along a spiral curve. The form of this spiral can easily be constructed by taking a strip of paper with a straight edge, and wrapping it round a pencil in a slanting direction; the edge forms a curve like that along which the thread or groove runs.

The screw works in a *collar* or *nut* **C,** through which a hole is bored, having a groove to fit the thread or a thread to fit the groove of the screw.

When the screw turns in a fixed collar, it moves forward in the direction of its length. In each turn of the screw, the distance moved forward is equal to the distance between consecutive threads: *i.e.* the distance **DE** between two consecutive turns of the thread, measured along the length of the bolt. This distance is called the **step.** Hence, by turning the screw round, it may be used to raise weights or overcome resistances applied to its end.

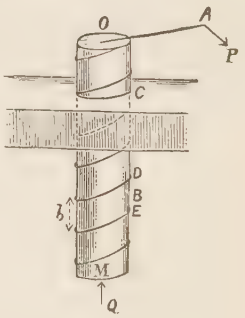


Fig. 112.

The *effort* must tend to turn the screw, and must therefore have a *moment* about **OM** in a plane perpendicular to **OM.** Hence the effort may be a single force *P* applied at the end of a long arm **OA,** projecting at right angles to **OM.** More often the arm projects in both directions, as in the common screw-press of Fig. 113, and two equal and opposite forces constituting

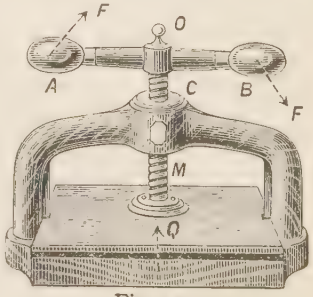


Fig. 113.

a *couple* are then applied perpendicularly to its two extremities **A, B.**

132. To find the mechanical advantage of a screw working without friction.

Let the effort P be applied perpendicularly at the end of an arm **OA** of length a, and let it overcome a resistance Q acting along the axis **OM**. Let b be the "step" or distance between two consecutive threads. Then, if the screw makes one complete turn, **A** the point of application of P will describe a circle of radius a about **O**, and will therefore move through a distance $2\pi a$ in the direction of P.* Also the screw will move through a distance b against the resistance Q.

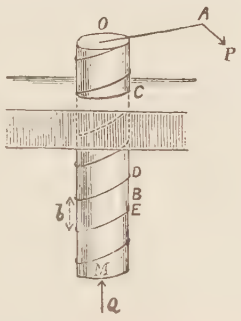


Fig. 114.

Therefore, since

work done by P = work done against Q,

we have $P \times 2\pi a = Q \times b$.

$$\therefore \text{ mechanical advantage } \frac{Q}{P} = \frac{2\pi a}{b} \quad \ldots\ldots\ldots\ldots\ldots (7)$$

$$= \frac{\text{circumference of circle described by the arm}}{\text{step of screw}}.$$

Cor. 1. Since the moment of P is $P \times a$, we have

$$\text{moment of } P = Q \times \frac{b}{2\pi}.$$

* As the screw moves forward along its axis **OM**, **A** really describes a spiral and not a circle; but, since P acts perpendicular to **OM**, the forward motion contributes nothing to the work done by P, which is therefore $P \times 2\pi a$.

If the quantity $b/2\pi$ be called the **pitch** of the screw* then **moment of effort = (resistance) × (pitch)** .....(8).

Cor. 2. By the Principle of Moments the effort P may be replaced by any other force or couple whose moment tending to turn the screw is the same. If two forces F act in opposite directions at the ends of an arm **AB** of length $2a$ fixed at its middle point to the screw, the moment of the couple thus formed $= 2Fa$, and therefore

$$2Fa = Q \times b/2\pi,$$

a relation easily verified by the Principle of Work. This shows that the effect is the same as if the two forces F were both applied at one end of the arm, forming a single force $2F$.

Example.—A screw-press is turned by applying two forces of 21 lb. in opposite directions at the ends of an arm 2 ft. long. If the step is $\frac{1}{8}$ in., to find (i.) the resistance overcome, (ii.) the pressure in lb. per sq. in. produced over the area of a circular piston 1 ft. in diameter.

(i.) Let Q be the resistance. Then in one revolution of the screw the works done by the efforts and against Q are

$$2 \times 21 \times 2\pi \,.\, 1 \text{ ft.-lb.} \quad \text{and} \quad Q \times \tfrac{1}{8} \times \tfrac{1}{12} \text{ ft.-lb.}$$

$$\therefore \quad \tfrac{1}{96}Q = 84\pi, \quad \text{or} \quad Q = 96 \times 84\pi.$$

Taking $\pi = \frac{22}{7}$, this gives $Q = 25344$ lb. $= 11\frac{1}{3}$ tons, roughly.

(ii.) Also area of piston $= \pi(\text{radius})^2 = \pi \,.\, 6^2 = 36\pi$ sq. in.

$$\therefore \quad \text{required pressure} = \frac{Q}{36\pi} = \frac{96 \times 84}{36} = 224 \text{ lb. per sq. in.}$$

$$= 2 \text{ cwt. per sq. in.}$$

132*a*. **The Differential Screw.**—By the following modification of the screw-press we can obtain a very great mechanical advantage (Fig. 114*a*.)

The screw **M**, which is turned by the lever, works as usual in a *collar* or *nut* in the framework at **C**. To the top of the press **A** is fastened another screw **N** of smaller diameter than **M**. The screw **N** works in a nut which is cut in the lower extremity of the screw **M**, and both **M** and **N** are "right-handed" screws. Also *b* the "step" of the screw **M** is slightly larger than *c* the step of the screw **N**.

* This is the definition of "pitch" used by Sir Robert Ball in his "Theory of Screws," and now generally adopted, although in a number of books before us we find no less than *three* different definitions of *pitch*.

Now let P be the force applied at the end of the lever whose length is a, and let Q be the resistance of the press. Then, as before, the work done by P in one revolution is $P \times 2\pi a$. Again in one revolution the screw **M** travels downwards through a distance b while the screw **N** penetrates a distance c into the screw **M**. Hence the press **A** is only forced down through a distance $(b - c)$, and the work done against Q is $Q \times (b - c)$.

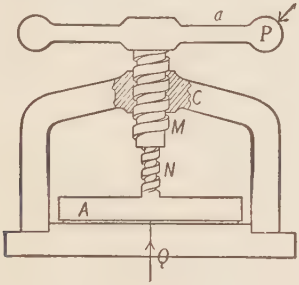


Fig. 114*a*.

Hence from the equation of work, we have

$$P \times 2\pi a = Q \times (b - c),$$

i.e., **mechanical advantage** $\dfrac{Q}{P} = \dfrac{2\pi a}{b - c}$ ..... (9)

Since c is very nearly equal to b, the denominator of this fraction is very small, and the mechanical advantage is very great.

132*b*. **The Differential Pulley.**—In this machine the rope is replaced by a long chain, of the same type as a bicycle chain. The upper pulley in the accompanying illustration (Fig. 114*b*.) is the differential pulley, and consists essentially of two concentric pulleys (of different radii a and b) cut in one block; but the pulley grooves are replaced by cogs on to which the links of the chain fit. The figure shows how the chain passes round the pulleys.

If the differential pulley makes one revolution the length of chain which passes over the larger circumference of the

differential pulley is $2\pi a$, and the length wound over the smaller circumference is $2\pi b$. Hence the total length of chain connecting the upper and lower pulleys is diminished by $2\pi(a - b)$. But this decrease is divided between the two vertical lengths of chain; thus the weight Q is raised a vertical height equal to the half of this decrease, or $\pi(a - b)$.

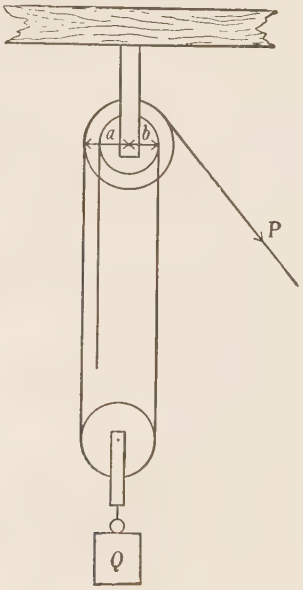


Fig. 114*b*.

Therefore the work done on Q is $Q \times \pi(a - b)$.

Also the work done by P is obviously $P \times 2\pi a$.

Equating these two amounts of work we have

$$P \times 2\pi a = Q \times \pi(a - b),$$

whence, **mechanical advantage** $\frac{Q}{P} = \frac{2a}{a - b}$ ..... (10).

The mechanical advantage may be made very great by making b nearly equal to a.

132*c*. **The Barton.**—The figure shows a system of pulleys known as the Barton (Fig. 114*c*.)

The effort P will determine the tension of the whole string to which it is applied. Let T be the tension throughout the string which passes over the fixed pulley.

From the equilibrium of the moveable pulley on the left, we obtain

$$T = 2P;$$

and from the equilibrium of the moveable pulley on the right,

$$2P + T = Q.$$

Whence

$$Q = 4P,$$

i.e. the system gives a mechanical advantage of 4.

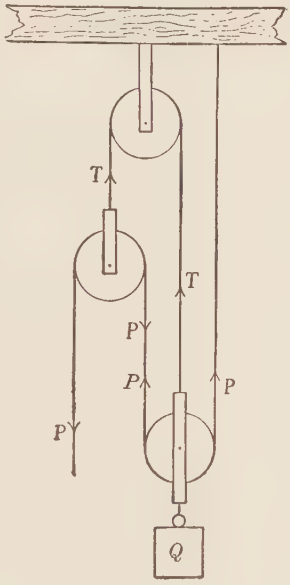


Fig. 114*c*.

NOTE.—It is easy to apply the Principle of Work to this machine. If the right pulley rises 1 ft., the left pulley falls 1 ft.; and it then follows that the point of application of the effort descends 4 ft.; whence the mechanical advantage is 4.

SUMMARY OF RESULTS.

In *fixed* pulley,

$W = P$, mech. advantage $M = 1$ ....(1). (§ 116.)

With a *single moveable pulley*.

$Q = 2P$, $M = 2$ (pulley light) ....(2), (§ 117.)

$W + w = 2P$ (pulley heavy) ....(2*a*). (§ 118.)

The "*second*" system is the single string system, pulleys in two blocks. With n pulleys and string hanging from upper block,

$Q = nP$, $M = n$ (pulleys light) .. (3), (§ 119.)

$W + w = nP$ (pulleys heavy .. (3*a*). (§ 120.)

The "*first*" is the separate string system, with strings tied up to fixed support. With n moveable pulleys,

$Q = 2^n P$, $M = 2^n$ (pulleys light) .. (4). (§ 121.)

$W + w_1 + 2w_2 + \ldots + 2^{n-1} w_n = 2^n P$ (pulleys heavy, lowest w_1) .. (4*b*). (§ 129.)

The "*third*" is the previous system turned upside down, strings all attached to weight.

$Q = 2^n P - P = (2^n - 1)\, P$, $M = 2^n - 1$ (pulleys light) .. (5). (§ 123.)

$W = (2^n - 1)\, P + w_2 + 3w_3 + \ldots (2^{n-1} - 1)\, w_n$ (pulleys heavy, lowest w_n) .. (5*a*). (§ 130.)

For man pulling himself up, mechanical advantage is increased by unity ..... (6). (§ 126.)

For a *smooth screw*, mechanical advantage

$$M \text{ or } \frac{Q}{P} = \frac{2\pi a}{b} = \frac{\text{circumference of circle described by } P}{\text{distance between two threads}} \quad .. (7). \quad (\S 132.)$$

$$\frac{\text{moment of } P}{Q} = \frac{b}{2\pi} = \frac{\text{step}}{2\pi} = \text{pitch of screw} \ldots . \quad (8).$$

For the *Differential Screw*,

$$\frac{Q}{P} = \frac{2\pi a}{b - c} \quad \ldots\ldots\ldots \quad (9). \quad (\S 132a.)$$

For the *Differential Pulley*,

$$\frac{Q}{P} = \frac{2a}{a - b} \quad \ldots\ldots\ldots \quad (10). \quad (\S 132b.)$$

For the *Barton*, $Q = 4P$ ..... (§ 132*c*.)

EXAMPLES IX.

(*Selected Problems* :—1, 2, 3, 7, 8, 11, 13, 15, 17, 18, 20, 22, 23, 26, 30, 31, 32.)

1. Find the ratio of the power to the weight in that system of pulleys in which each moveable pulley hangs by a separate string, the number of moveable pulleys being 4.

If the weights of the pulleys be taken into account, and are 1, 2, 3, and 4 lb., respectively, beginning with the highest, find what power will support a weight of 294 lb.

2. Find the ratio of the power to the weight in that system of pulleys in which each string is attached to the weight, the number of moveable pulleys being 4.

If the weights of the moveable pulleys be taken into account, and are 1, 2, 3, and 4 lb., respectively, beginning with the lowest, find what power will support a weight of $4\frac{1}{4}$ cwt.

3. If there are 4 pulleys in the third system, and each weighs 2 lb., what weight can be raised by a power equal to the weight of 20 lb. ?

4. A man weighing 15 stone, holding the bar to which the strings are attached in the third system of pulleys, can just raise a child weighing 1 stone, holding on to the last string. How may pulleys are there ?

5. If there be three moveable pulleys, each of mass 1 lb., in a system in which all the strings are parallel and are attached to the weight, and the force required to support a certain weight is half that which would be required if the pulleys were weightless, find that weight.

6. Is it more advantageous in raising a weight by (i.) the first system, (ii.) the third system, of pulleys, to have the pulleys heavy or light ?

7. If in a system of pulleys in which each hangs by a separate string there be three pulleys, each of mass 1 lb., and the power required to support a certain weight is twice that which would be required if these pulleys were weightless, find that weight.

8. Being given four weightless pulleys, find the greatest weights that can be supported by a force of 1 lb. when the pulleys are arranged as in the first, second, and third systems, respectively, in such a way that the effort is applied *downwards*.

9. If in Ex. 8 the effort of 1 lb. had to be applied *upwards*, what would be the weights lifted in the corresponding arrangements?

10. In the first system of pulleys, find from first principles the power necessary to support a weight of 4000 lb. when there are four moveable pulleys. Find also the pull of each rope on the beam; and, if the sum is not equal to the weight, explain the difference.

11. If a man whose weight is 10 stone supports 2 cwt. by a block-and-tackle, there being 3 pulleys in the block, what is the pressure on the floor on which the man stands, and what the pressure on the beam to which the upper pulley-block is attached?

12. In the system of pulleys in which each pulley hangs by a separate string, how would you find experimentally the relation between the tension of any string and that of the string next above it?

13. Two bodies connected by means of a string passing over a smooth pulley touch each other at one point; show that the stress between them cannot be horizontal unless their weights are equal.

14. A man weighing 10 stone supports a weight of 91 lb. by means of 3 moveable pulleys arranged in the first system and weighing, respectively, 2 lb., 4 lb., 5 lb. What is the thrust of the man on the ground?

15. Apply the Principle of Energy to prove that in a system consisting of three very light pulleys, of which the first is fixed and each of the others is supported by the string of the preceding one, the strings being all parallel and all attached to the weight W, the downward acceleration of this weight, when the power of P is not sufficient to balance it, is $\frac{W - 7P}{W} g$.

16. In the second system of pulleys, what must be the relation between the radii of the pulleys at the lower block in order that they may all be grooved in the same piece?

17. In the third system of pulleys, a weight of 45 lb. is supported by a force of 3 lb. weight. The strings are attached to a 6-ft. bar from which the weight is suspended, and at distances of 1 ft. from each other. To what point of the bar should the weight be attached? The bar and pulleys may be considered weightless.

18. The distance between two consecutive threads of a screw is $\frac{1}{4}$ in., and the length of the power arm is 5 ft. What weight will be sustained by a power of 1 lb.?

19. If a power of 10 lb. acting on an arm 2 ft. long produces in the screw press a thrust of one ton weight, what is the "step" of the screw?

20. The arm of a screw-jack is 2 ft. long, and the screw rises 2 in. when it is turned round nine times. What force must be applied to produce a thrust of a half a ton weight?

21. A screw is formed upon a cylinder whose length is 1 ft. and circumference 3 in. How many turns must be given to the thread in order that a power of 4 lb. weight, applied tangentially at the edge of the cylinder, may support a weight of 2 cwt.?

22. A weight W is supported by a small frictionless pulley which runs on a string of length l, whose extremities are attached to two fixed points **P** and **Q**. Given that **PQ** $= d$, and that the inclination of **PQ** to the horizon is α, find the tension of the string.

23. In the second system of pulleys, if the total number of pulleys is four, and the weight at the lower block is 16 lb., and the mass of the pulleys is negligible, find the upward acceleration of the lower block, (*a*) if the cord is pulled with a force of 5 lb. wt., (*b*) if a weight of 5 lb. is attached to the free end of the cord.

24. In a wheel and axle where the mass of the machine is negligible a weight of 12 lb. is balanced by a force of 3 lb. Find the acceleration of the weight, (*a*) if the cord is pulled with a force of 4 lb. wt., (*b*) if a weight of 4 lb. is attached to the free end of the cord.

25. In any machine if a weight of W lb. is balanced by a force of P lb. wt., and if the mass of all the moveable parts of the machine can be neglected, find the acceleration of W, (*a*) if the power is increased to Q lb. wt., (*b*) if the power is replaced by a weight of Q lb.

26. A "differential" screw consists of two screws formed at the two ends of the same cylindrical rod, worked by an arm at the centre. If it is used to draw two blocks together the effect of turning the arm is that the one screw works its way into a worm connected with one block, and the other works more slowly out of a worm connected with the other block. If the step of the first screw is a, and of the second b, and the arm is of length r, find the mechanical advantage.

27. If the third system of pulleys is modified by making the string that passes over each pulley **A** pass round a small pulley attached to the weight, and the string is then fastened to the pulley **A**, show that if the weights of the pulleys be neglected, and the number of separate strings be n, the mechanical advantage will be $3^n - 1$.

28. In a system of n weightless pulleys, each hanging by separate strings whose fixed ends are parallel and make an angle θ with the vertical, show that the mechanical advantage is

$$\sin 2^n\theta / \sin \theta$$

29. A weight is supported as follows: n moveable pulleys and a fixed one are arranged as in the third system (in which the strings are all attached to the weight), and the string which passes over the lowest pulley of the system passes round a pulley fixed to the weight, and is then attached to the lowest of a set of n moveable pulleys arranged as in the first system (in which the strings are all attached to the beam). All the strings being vertical and all the pulleys of weight w, find the power required to raise a weight W.

30. Prove that a force of 14 lb. applied at the end of a lever 3 ft. long, to tighten up a nut on a screw-bolt, one inch in diameter, and with 8 threads to the inch, will be sufficient to break the bolt if the breaking tension of the material is less than 14·4 tons per square inch.

31. If in the second system of pulleys the working tension of a rope is reduced by the factor f in passing over a pulley (in consequence of resistances due to friction, etc.); prove that the mechanical advantage is $(f - f^{n+1})/(1 - f)$.

32. Find the mechanical advantage of the differential pulley without using the principle of work.

EXAMINATION PAPER IV

1. Classify the different kinds of levers, pointing out those in which there is a mechanical advantage.

2. Explain how the mechanical advantage of an oar will be altered by altering the position of the point in contact with the rowlock.

3. "What is gained in power is lost in speed."—Explain clearly what is meant by this statement, and show that it is true in the case of the wheel and axle.

4. In a wheel and axle the radius of the wheel is 2 ft. and that of the axle is 2 in. What power will balance a weight of 97 lb. if the ropes to which the power and weight are applied are each $\frac{1}{2}$ in. in diameter?

5. State the Principle of Work, and apply it to find the relation between the effort and weight in the single-string system of pulleys, the strings being parallel and the number of pulleys n.

6. A straight lever has equal arms, and a given force acts at one extremity at an angle θ to the lever. At what angle with the lever must the balancing force act at the other extremity in order to produce the minimum pressure on the fulcrum?

7. In a screw-press a power equal to 16 lb. weight, acting on an arm 3 ft. long, produces a thrust of half-a-ton. What is (i.) the "step," (ii.) the pitch, of the screw?

8. In the second system of pulleys, if a weight of 3 lb. supports a weight of 15 lb., and a weight of 5 lb. supports a weight of 27 lb., what is the weight of the lower block, and what would the mechanical advantage be if the lower block were weightless?

9. **A** and **B** are two points in a horizontal line; **AC** is a uniform rod of weight W freely hinged at **A**, and supported in a sloping position by a string **CB** fastened at **B**. If the angles **BAC** and **ABC** are respectively α and β, find the tension of the string.

PART III.

CENTRES OF GRAVITY.

CHAPTER X.

CENTRES OF PARALLEL FORCES. DEFINITIONS.

133. We have frequently assumed the fact that the weight of a heavy straight uniform rod or beam may be supposed to act at its middle point, and, generally, that the weight of a rigid body of any shape may be supposed to be concentrated at a single point, called the centre of gravity of the body. We shall now prove this property, and in Chap. XII. we shall show how to determine the position of this point for bodies of certain shapes.

We commence by proving a few further theorems about parallel forces.

134. To find the resultant of any number of parallel forces of given magnitudes applied at given points of a rigid body, not necessarily in the same plane

Let the given parallel forces be P acting at **A**, Q at **B**, R at **C**, S at **D**, and so on.

Join **AB**, and in **AB** take a point **E** such that

$$P \times \mathbf{AE} = Q \times \mathbf{EB}.$$

Then, by § 77, the forces P and Q are equivalent to a single resultant force $P + Q$ acting at **E**, parallel to both of them.

Now compound this resultant with the force R at **C**. Join **EC**, and on it take a point **F**, such that

$$(P + Q) \times \mathbf{EF} = R \times \mathbf{FC};$$

then the forces $P + Q$ at **E** and R at **C** are equivalent to a single resultant force $P + Q + R$ acting at **F** parallel to them.

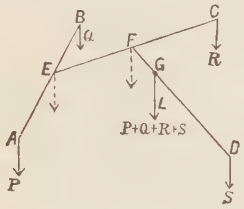


Fig. 115.

This resultant is therefore the resultant of the *three* forces P, Q, R at **A, B, C,** respectively.

Now join **FD**, and in it take a point **G**, such that

$$(P + Q + R) \times \mathbf{FG} = S \times \mathbf{GD};$$

then the forces $P + Q + R$ at **F** and S at **D** are equivalent to a single resultant $P + Q + R + S$ acting at **G** parallel to them.

This resultant is therefore the resultant of the *four* forces, P, Q, R, S, at **A, B, C, D,** respectively.

Proceeding in this way, we may find the resultant of any number of parallel forces acting at different points of a body.

We might have compounded the four forces P, Q, R, S, in any other order whatever. Thus, if we had first found the resultant of R and S, then compounded this with Q, and compounded the resultant so obtained with P, we should have a different construction for the resultant of the four forces. The point **G** finally arrived at will be the same in every case.

135. The Centre of Parallel Forces.—DEFINITION.—The above constructions show that—

The resultant of any number of parallel forces passes through a certain point whose position depends only on the magnitudes and the points of application of the forces and not on their direction.

This point is called the **centre** *of the parallel forces.*

Thus, in § 134, **E** is the centre of P and Q, **F** is the centre of P, Q, R, and **G** is the centre of P, Q, R, S.

If the forces P, Q, R, S, instead of acting as in Fig. 115, are applied at the same points **A, B, C, D**, but in a different

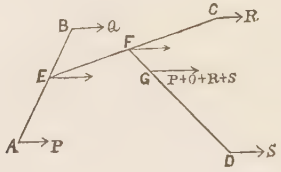


Fig. 116.

direction (though still parallel), as in Fig. 116, their resultant will still pass through the same point **G** as before. And this can be said of no other point.

OBSERVATION.—The student should carefully notice the difference between the *point of application of the resultant* and the *centre* of a system of parallel forces.

If P, Q, R, S act at **A, B, C, D** *in a given direction,* their resultant acts in a straight line **GL**, *parallel to this direction and passing through the centre* **G**. The resultant *may be supposed to act at* **G**, but it *need not necessarily be applied* at **G**; for, by the Principle of Transmission of Force, it may be applied at any point (say **L**) in its line of action (Fig. 115). But let P, Q, R, S, still acting at **A, B, C, D**, be turned round into a different direction. Their resultant will still pass through **G**, the centre of the forces, but it will no longer pass through **L** (Fig. 116).

136. **A system of parallel forces cannot have more than one centre.**—For the resultant of several parallel forces

cannot pass through two points except when the forces are parallel to the line joining the points.

137. **Centre of parallel forces acting at points in a straight line.**—If a number of parallel forces P_1, P_2, P_3 are applied at points $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$ lying alongside a straight rod or beam, the construction for the centre of parallel forces (§ 134) shows that their centre also lies in the straight line through $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$. If x_1, x_2, x_3 are the distances of $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$ from a fixed point **O** on the rod, the distance of the centre from **O** is therefore given by the formula of § 84,

$$\bar{x} = \frac{P_1x_1 + P_2x_2 + P_3x_3 + \ldots}{P_1 + P_2 + P_3 + \ldots}.$$

This follows at once, by taking moments, if we suppose the forces applied perpendicular to the rod. If they act in any other direction, the products P_1x_1, P_2x_2, P_3x_3, ..will not be the moments of the forces about **O**; but, since the centre of the parallel forces is unaltered in position, the formula shows that it may be found *by treating the products* P_1x_1, P_2x_2, P_3x_3, ..*as if they were moments.*

138. **On the centre of gravity.**—The "weight" of a body is not a single force, for each particle of the body has its own weight. The weights of the particles form a system of like parallel forces; for the weight of each particle acts vertically down. But we may suppose this system of parallel forces replaced by its resultant, and it is convenient to speak of this resultant as the "weight of the body."

The weight of a single particle is not really a single force, for each particle of the earth exerts its own attraction on the given particle, and the "weight" of the particle is the resultant of all such attractions.

Now suppose that we consider three particles **A, B, C** of a body, whose weights are respectively W_1, W_2, W_3; and suppose that the body is rigid, *i.e.* it can be placed in different positions (as in the diagrams of Fig. 116*a*) without altering its shape or size.

Then in either position, the resultant of the weights of the particles **A, B** is a force $(W_1 + W_2)$ acting at a point **D**,

which divides **AB** in the ratio $W_2 : W_1$; and since the positions of **A** and **B** are *fixed with regard to the body*, **D** will also be *fixed with regard to the body*. Again, combining the force at **D** with W_3 at **C**, we see that in either position of the body the resultant weight of the three particles **A, B, C** is a force $(W_1 + W_2 + W_3)$ acting at a point **E** which divides **CD** in the ratio $W_1 + W_2 : W_3$; and we see also that the position of **E** is fixed with regard to the body.

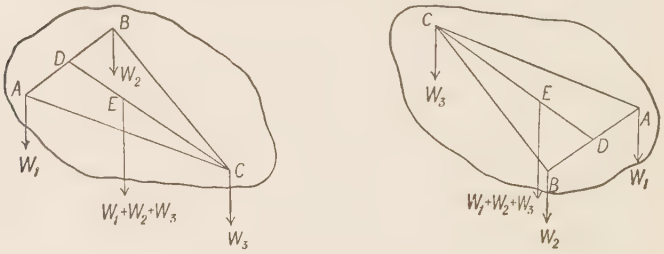


Fig. 116*a*.

By extending this process till we have included all the particles of the body (in either position) we can find the point of application of the resultant weight of the whole body; and it will follow that, provided the body does not alter its shape or size, the point of application of the resultant weight is fixed with regard to the body, or (as it is sometimes expressed) "moves with the body." This point is called the **centre of gravity** of the body.

The centre of gravity need not be in the body itself; thus the centre of gravity of a circular ring of wire is at the centre of the circle and not in the material of the wire itself.

The above theorem is sometimes quoted in the form "the centre of gravity of a rigid body is fixed relative to the body." We define a rigid body as one whose size and shape cannot be altered, or more accurately as *a body such that the distance between any two of its particles is unaltered, whatever forces may be applied to the body*. Of course no body is perfectly rigid, but for ordinary purposes of calculation many solid bodies may be considered rigid.

It is not necessary that the body itself should be rigid in order to have a centre of gravity, but the centre of gravity remains fixed relative to the body only as long as its size and shape remain unaltered.

Thus a straight piece of wire has a centre of gravity at its middle point, and its weight will act at that point as long as it remains straight. If the wire be bent, it will no longer have the same centre of gravity. A bicycle has a centre of gravity through which its weight will always act, so long as its different parts (the steering-wheel and handle, and the frame, etc.) preserve the same relative position.

139. We may therefore give the following :—

DEFINITION.—*The* **centre of gravity** *of a body is that point, fixed relative to the body, through which the resultant force due to the Earth's attraction on it always passes, whatever be the position of the body, provided that its size and shape remain constant.*

In short, *the centre of gravity is the point at which the whole* **weight** *of the body may always be supposed to act.*

The abbreviation for centre of gravity is **C.G.**

140. **The construction for the C.G. of a system of particles** at given points is identical with that for the centre of parallel forces, the weights of the particles being the parallel forces. It is, therefore, given by § 134 or § 137.

141. **A body cannot have more than one C.G.**—For if the body had two centres of gravity **G, H,** the line of action of the body's weight would always have to pass through both **G** and **H,** and would therefore have to be the line **GH**. But this is impossible, except when **GH** is vertical, for the weight of a body always acts vertically. Hence the body cannot have two C.G.'s.

[*Compare this proof with* § 136.]

142. **Centre of mass.**—The foregoing considerations show that a body has a C.G. only when—

(i.) It is acted on by gravity, *i.e.* it has *weight*.

(ii.) The intensity of gravity is constant, both in magnitude and direction, at all points of the body (as is the case for bodies of small size as compared with the Earth).

Now, if gravity were not to exist, a body would not have *weight*, but it would still have what is called *mass*, although its mass could not be measured in the ordinary way by *weighing*, and the analogy naturally suggests (as we shall now learn) that the body would still have a *centre of mass*.

Suppose a body to be at first acted on by gravity, and let its C.G. be found and indicated in position by marks made on the body. Suppose the body to be then removed from the influence of gravity by being carried right away from the Earth into space. The marks will still indicate the same point of the body, and the position of this point will have a definite relation to the distribution of matter in the body, although it can no longer be called its centre of *gravity*. This point will be called the body's *centre of mass*. It may be defined *statically* as follows :—

143. DEFINITION.—The **centre of mass,** or **mass-centre,** of a body is the centre of a system of parallel forces acting on all the particles into which the body may be supposed to be divided, the force on each particle being proportional to its mass.

The abbreviation for centre of mass is **C.M.**

We observe that every body, or system of bodies, has always a C.M.* *If* it have a C.G., that point will *necessarily* coincide with its C.M., because the weights of the particles of the body, being proportional to their masses, form a system of forces having the C.M. for their centre.

It is, therefore, always correct to speak of the C.G. of a body as its centre of mass, but not *vice versâ*.

As we are only concerned with the effects of terrestrial

* For the definition makes no assumption as to the *cause* producing the forces on the particles of the body, and only assumes that these particles have *mass*.

gravity, we shall continue to use the term C.G. By finding the C.G., we really determine the C.M. ; just as by *weighing* a body we really determine its *mass*.

[* 144. The following *dynamical* definition may be substituted for the above equivalent statical definition :—

DEFINITION.—The **centre of mass** of a body is the point through which the resultant (" impressed ") force on the body acts when all parts of the body move with the same acceleration.*

For the forces acting on different parts of the body are proportional to their respective masses.]

145. **Construction for the C.M. of a number of particles.** —If particles, whose *masses* are $P, Q, R, S, \ldots$, are situated at points **A, B, C, D,** the point **G,** obtained by the construction of § 134, will be the centre of parallel forces of magnitudes P, Q, R, S acting at **A, B, C, D,** respectively, and will therefore be the C.M. of the particles.

Similarly for particles in a straight line the C.M. is given by the formula of § 137.

146. DEFINITIONS.—A **lamina** is a sheet of material whose thickness is so small that it may be regarded as a distribution of matter over an area. A **uniform** lamina is one which is of the same thickness and formed of the same substance throughout.

The C.G. of an area or surface means the C.G. of a uniform lamina covering that area or surface.

Thus a sheet of paper or thin card, a thin sheet of metal such as that forming a tin canister, a plate of glass of small thickness, a slate, and in some cases even a wooden plank, may be regarded as a lamina. And the C.G. of a parallelogram will be the C.G. of a sheet of paper or any other uniform lamina forming that parallelogram.

A wire, stick, rod, or beam is said to be **uniform** when it is of the same cross section and of the same substance throughout its length.

The C.G. of a line (whether straight or curved) means the C.G. of a thin uniform wire placed along that line.

* This is the case when the motion of the body is one of " pure translation."

147. The C.G. of a straight line is at its middle point.

This is obvious, for there is no reason why it should be nearer one end than the other.

To prove it, let **AB** be a straight uniform wire, **G** its C.G. Then **G** evidently lies somewhere in the line **AB**. And if the wire be turned round so that the end **B** is placed at **A** and the end **A** is placed at **B**, the wire lies along the same line as before, and the position of **G** must therefore be unaltered. Hence **GA** = **GB**, or **G** is the middle point of **AB**.

Fig. 117.

148. To find the C.G. of a parallelogram.

Let **ABCD** be *either* a uniform lamina or a uniform wire in the shape of a parallelogram. Let its diagonals **AC, BD** intersect in **G**. Then shall **G** be the required C.G.

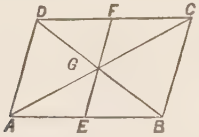


Fig. 118.

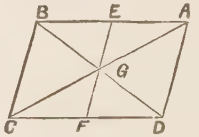


Fig. 119.

Turn the parallelogram round as in Fig. 119, and place it so that the vertices **A, B, C, D** coincide with the previous positions of the vertices **C, D, A, B**, respectively. Then the diagonal **AC** will coincide with its previous position **CA**, and **BD** with **DB**. Hence **G**, the intersection of the diagonals, will be unaltered in position. Also no other point in the parallelogram will occupy the same position as before. For any other point will be brought round to the opposite side of **G**.

Now, when the parallelogram is turned round, it occupies the same space as before; hence its C.G. in its new position coincides with its C.G. in its old position. Therefore the C.G. of the parallelogram must be at **G**.

OBSERVATION.—The reader should cut the parallelogram out of a sheet of paper, marking the points **A, B, C, D** both on the parallelo-

gram and on the paper from which it was cut. On turning it round into the new position, it will again exactly fill the hole from which it was cut, and the diagonals will be unaltered in position, but their extremities will be interchanged. Hence **G** will be the intersection of the same two straight lines as before, and will therefore be unaltered in position.

Cor. 1. Since the diagonals of a parallelogram bisect each other, *the* c.g. *is at the middle point of either diagonal.*

This is also obvious from the method of proof. When the parallelogram is turned round, **AG** is brought into coincidence with the former position of **GC**. Therefore **AG** = **GC**, and **G** is the middle point of **AC**.

Cor. 2. *The* c.g. *is the middle point of the line bisecting a pair of opposite sides.*

For let **E, F** be the middle points of **AB, CD**. When the parallelogram is turned round, **AB, CD** occupy the former places of **CD, AB**, respectively. Hence **E, F** occupy the former places of **F, E**, respectively; and therefore the middle point of **EF** occupies the same position as before. Therefore this point is **G**, the required c.g. of the parallelogram.

Cor. 3. Similarly, **G** is the middle point of the bisector of the pair of opposite sides **BC, DA**. *Hence the diagonals and the bisectors of the opposite sides of the parallelogram all bisect one another in the* c.g. *of the parallelogram*, as may be otherwise proved by geometry.

149. The C.G. of

(*a*) a circular ring, and

(*b*) a circular area,

are at the centre of the circle itself.

For, if either a uniform circular wire or a uniform circular lamina is rotated about its centre, it will continue to occupy the same space as before; therefore its c.g. will be unaltered in position. But if the c.g. were not at the centre of the circle, its position would rotate with the wire or lamina, and would therefore change. Therefore the c.g. cannot be elsewhere than at the centre (as is almost obvious without proof).

150. Other symmetrical figures.—The method of the last three articles depends on the fact of the figure whose C.G. is required being "**symmetrical** about a point," in such a way that it is *capable of being turned about that point till it occupies exactly the same space as before, without its different parts all coming round to their original positions.* We find, in each case, that *one* point alone remains unchanged in position, and we conclude that this point must be the required C.G.

A body of *any* shape could, of course, be *rotated completely round* (*i.e.*, turned through four right angles) about any point and brought back into its original position ; but this would *not* help us to find the C.G., because *every* point of the body would come back to its former position, so that we should be no nearer knowing *which* was the C G. than we were before.

From the above property we are able at once to write down the following additional results :—

The C.G. of the area of a **regular polygon** is the centre of the polygon, and the same is true for a uniform wire bent round to form the sides or " perimeter " of a regular polygon. For the C.G. must be unaltered in position when the polygon is turned about the centre, and each side is brought into the former place of the next side.

The C.G. of a solid **sphere,** or of the surface of a sphere, is at the centre of the sphere, for its position is evidently unaltered by turning the sphere round about its centre through any angle.

The C.G. of the volume or the surface of a **right cylinder** is at the middle point of its axis, for it is evidently unaltered in position when the cylinder is revolved about its axis, and also when turned, so that the positions of the two ends are interchanged.

The C.G. of the volume or surface of a **cube** or any other **parallelepiped** (*e.g.*, a brick) is at a point midway between each pair of opposite faces, as may be seen by turning it round so as to interchange the positions of two opposite faces. It may be proved geometrically that the diagonals all intersect and bisect one another at the C.G.

Example.—The sides **AB, BC, CA** of a triangle measure 4, 5, 3 inches respectively, and **AD** is drawn perpendicular to **BC**. Three weights are placed at **A, B, C** respectively, and their C.G. bisects **AD**. Find the ratios of the weights.

$$BC^2 = BA^2 + AC^2;$$

$$\therefore \quad \angle BAC = 90°.$$

Hence $\triangle$s **BDA, BAC** are similar.

$$\therefore \quad BD : BA = BA : BC; \text{ whence } BD = 3{\cdot}2.$$

Thus $CD = 1{\cdot}8$.

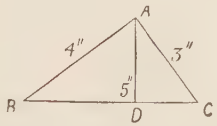


Fig. 119*a*.

Let the weights at **A, B, C** be x, y, z respectively. Then the C.G. of y and z must fall at **D**; for if it fell at some other point **E** in **BC**, the C.G. of x, y, z would fall in **AE**, and not in **AD**. Thus

$$y : z = CD : BD = 1{\cdot}8 : 3{\cdot}2 = 9 : 16.$$

Again the C.G. of $y + z$ at **D** and x at **A** bisects **AD**. Hence $y + z = x$.

Thus $\quad x : y : z = y + z : y : z = 25 : 9 : 16.$

Summary of Results.

The centre of parallel forces P, Q, R, S at **A, B. C, D,** etc. is found by dividing

AB at **E**, where $P \times AE = Q \times EB$;

EC at **F**, where $(P + Q)\ EF = R \times FC$;

FD at **G**, where $(P + Q + R)\ FG = S \times GD$;

and so on. (§§ 134, 135.)

The centre of gravity of a body is that point through which the resultant force due to the Earth's attraction always passes. (§ 139.)

The centre of gravity of

(1) *a straight line* is its middle point; (§ 147.)

(2) *a parallelogram* is the intersection of its diagonals; (§ 148.)

(3) *a circular ring* is the centre of the circle ; (§ 149.)
(4) *a circular area* is the centre of the circle ; (§ 149.)
(5) *a regular polygon* is the centre of the polygon ; (§ 150.)
(6) *a sphere* is the centre of the sphere ; (§ 150.)
(7) *a right cylinder* is the middle point of its axis ; (§ 150.)
(8) *a cube* or *rectangular parallelepiped* is at the intersection of its diagonals. (§ 150.)

EXAMPLES X.

(*Selected Problems* :—1, 3, 5, 8, 9, 11, 12.)

1. Equal parallel forces act at the angular points of an equilateral triangle. Find their centre (i.) when the forces are like, (ii.) when one of them is unlike the other two.

2. P, Q, R are parallel forces acting in the same direction at the angular points of an equilateral triangle **ABC**. If $P = 2Q = 3R$, find the position of their centre.

3. In the preceding question, find the position of the centre of the forces if the direction of Q is reversed.

4. Parallel forces of 3, 5, 7 and 11 lb. act at points in a straight line at distances of 4 in. from each other. What is the distance of their centre from the 5 lb. force ?

5. A uniform straight lever of weight 10 lb. and length 6 ft. balances about a certain point when weights of 3 lb. and 5 lb. are attached to its extremities. How far must the point of support be moved so that the lever may balance when each of the weights is increased by 1 lb. ?

6. Four equal forces act at the corners of a square in the same direction along parallel lines. What change will be produced in the position of their centre if the direction of one of them is reversed ?

7. If a series of parallel forces can be reduced to a couple, what is the position of their centre ?

8. A light straight wire 3 inches long carries a mass of 3 oz. at one end, a mass of 4 oz. 2 inches from that end, and a mass of 5 oz. at the other end. Find the C.G. of these masses.

9. **ABC** is a triangle ; **D**, **E** are points in **BC**, **AC** respectively ; **BE**, **AD** intersect in **O**. Three masses at **A**, **B**, **C** have their C.G. at **O**. Find their ratios, given **AO** : **OD** = 3 : 4, **BD** : **DC** = 2 : 7.

Deduce the ratios **BO** : **OE**, **CE** : **EA**.

10. A rod **ABC**, 16 in. long, rests in a horizontal position upon two supports at **A** and **B** 1 ft. apart, and it is found that the least upward and the least downward force applied at **C** which would move the rod are 4 oz. and 5 oz. respectively. Find the weight of the rod and the position of its centre of gravity.

11. Equal masses are placed at the angular points of a regular pentagon. Show that their centre of gravity is at the centre of the circumscribing circle.

12. If three parallel forces P, Q, R act at **A**, **B**, **C**, the angular points of a triangle **ABC**, find the condition that their centre falls at (i) the centre of the inscribed circle ; (ii) the centre of the escribed circle opposite to **A** ; (iii) the centre of the circumscribed circle ; (iv) the orthocentre.

CHAPTER XI.

PROPERTIES OF THE CENTRE OF GRAVITY.

151. Equilibrium of a heavy body about a fixed point.

If a heavy body be at rest when supported at one point, and be not acted upon by forces other than gravity, the C.G. *and the point of support will be in the same vertical line.*

For the only forces acting on the body are –

(i.) The resultant force due to gravity. This is the weight of the body, and acts vertically through the C.G.

(ii.) The reaction at the point of support.

These forces are in equilibrium; therefore they must be equal and opposite and in the same straight line.

Therefore the point of support is in the vertical through the C.G.

Fig. 120.

This refers to a body either resting on a surface at a point, as in Fig. 120, or capable of rotation about a point, or suspended by a string as in Figs. 121, 122.

152. **The plumb-line,*** used by builders and others for finding the direction of the vertical, is a flexible string from which a heavy weight of lead, or **plummet,** hangs. When at rest, the string is vertical throughout its length, for the tension at any point has only to support the weight of the plummet, which acts *vertically* (§ 138). We may sometimes define the *vertical* line *at any point* as the direction of a plumb-line at that point.

153. **To find by experiment the C.G. of a lamina.**

The C.G. of a sheet of metal or any other material, or a wooden plank, or a wire of any shape, may be found in the following manner :—

Suspend the body by a string attached to any point **A**, and on it draw the vertical line **AD** through **A**. This line may be traced either by means of a plumb-line or by producing the direction of the string. We know, by § 151, that the C.G. lies in **AD**.

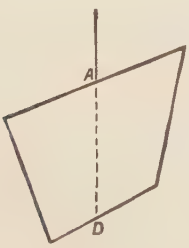


Fig. 121.

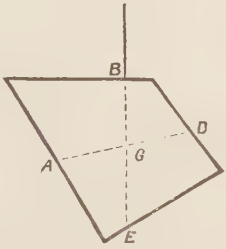


Fig. 122.

Now suspend the body from any other point **B**, and on it draw the vertical line **BE** through **B**. As before, the C.G. lies in **BE**. Therefore the required C.G. is at the intersection of **AD, BE**.

* From the Latin *plumbum* = " lead."

If we now hang the lamina from a third point **C**, it will be found that **G** is vertically below **C**, thus verifying that **G** is the C.G.

This method is *theoretically* applicable to any body whatever, but except in the case of a lamina, or a wire bent in one plane, it would usually not be very easy to mark the two positions of the vertical, **AD** and **BE**.

If the lamina is in the shape of a rectangle or parallelogram, and is hung up from one of its angles, the diagonal through that angle will be found to be vertical, thus verifying that the C.G. lies in the diagonals.

To apply the method to a straight thin rod (*e.g.* a ruler, a straight walking-stick, or better still, a billiard cue), we suspend it by means of two very fine strings from any point **O**. Also make a plumb-line by hanging a small weight **W** by a single thread from **O**. If the thread **OW** cuts the rod in **G**, the C.G. of the rod will be at **G**. Mark this point on the rod. Then, if the rod be laid on the finger or on any support touching it at **G**, it will balance.

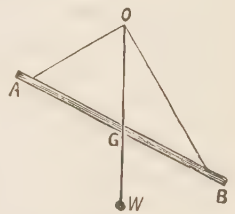


Fig. 123.

If the rod is uniform, **G** will be found to be at its middle point. If one end of the rod is heavier than the other, **G** will be found to be nearer to the heavier end.

Example.—A heavy rod **AB**, whose centre of gravity **G** is not at its middle point, is suspended by a single string **AOB** over a *smooth* peg **O**. Discuss the possible positions of equilibrium.

[The student should draw a figure very like Fig. 123, but making **AG** rather shorter, so that **OG** bisects the angle **AOB**.]

Since the weight of the rod at **G** acts vertically downwards, **OG** must be *vertical*.

Moreover, since the peg is smooth, the tension T is the same in each part of the string.

Hence the resultant of the two tensions T must bisect **AOB**, the angle between them. For equilibrium, this resultant is equal and opposite to the weight, and must act along **OG**. Thus **OG** must bisect the angle **AOB**, that is, *the strings must be equally inclined to the vertical.*

By Euclid VI. 3, $\frac{AO}{OB} = \frac{AG}{GB}$.

That is, *the parts of the string must be proportional to the segments of the rod.*

By drawing **GE** parallel to **BO**, we get a convenient triangle of forces, **OEG**, from which the relation between the tension and the weight can be calculated, in particular cases.

There are other less interesting positions of equilibrium, namely, with the rod vertical, and either end uppermost. If we assume the peg to be of insignificant dimensions, the three forces will then act in one straight line, and the tension will be equal to half the weight.

It is noteworthy that, in the case of a picture-frame as ordinarily suspended, the two parts of the string would be equally inclined to the vertical, even if the frame itself were hanging crookedly, if the peg were perfectly smooth.

154. If a heavy body is supported *at* its C.G., it will balance in every position.

Since the C.G. is at the point of support, the body's weight always acts through the point of support. This weight cannot move the point of support, because that point is fixed; and it cannot turn the body round, because it has no moment about that point. Hence the body must remain balanced in every position.

In consequence of this property, the C.G. *is very commonly spoken of as the* **balancing point** *of a body.*

155. DEFINITION.—When a body rests upon any hard flat surface, its **base** is defined to be the area enclosed by a fine string drawn tightly round it so as to enclose all points of the body in contact with the supporting surface.

When a glass tumbler rests on a table, the *base* of the tumbler is evidently the circular area enclosed by the parts of the tumbler touching the table. But in the case of a table resting on the floor on its four legs, the *base* is the quadrilateral formed by joining the feet, for this would be the figure assumed by a string pulled tightly round the points of contact. The definition ensures that when the body is overturned it must turn about some point or line bounding the base.

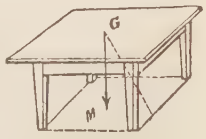


Fig. 124.

156. When a body is placed on a plane it will stand or fall according as the vertical line through its C.G. falls within or without the base on which it rests.

(i.) First let the vertical through the C.G. cut the supporting plane at a point **M** *outside* the base **AD**; let **A** be the point of the base that is nearest to **M**. Then the

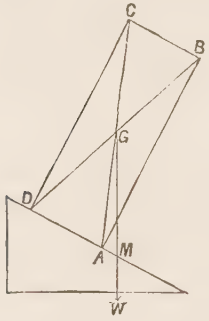


Fig. 125.

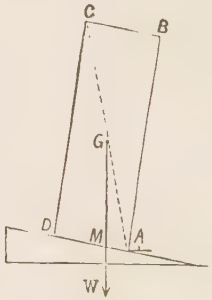


Fig. 126.

weight of the body is equivalent to a single resultant force acting at **G**, and the moment of this weight (Fig. 125) tends to turn the body about **A** in such a way as to lift up all the other points of the body touching the plane. Now the reaction of the plane always acts away from it, and never tends to prevent any part of the body from being lifted off. Hence there is nothing to counteract the tendency of the body to overturn about **A**. Therefore it will *fall*.

(ii.) Second, let the vertical through the C.G. cut the supporting plane at a point **M** inside the base **AD**. Then, if **A** be any point at the edge of the base, the moment of the weight about **A** tends to press down the other points of the body touching the plane (Fig. 126), as at **D**. Now the reaction of the plane prevents the body from penetrating it. Hence, the body will *stand*. And if it be

slightly tilted up, its weight will bring it back to its original position as soon as it is let go.

As a particular case, when a body rests touching the ground at a single point, the vertical through the centre of gravity must pass through that point (Fig. 120).

OBSERVATION.—The plane supporting the body may be either horizontal or inclined, provided that it is sufficiently rough to prevent the body from sliding down.

157. *Illustrations.*—(1) A cart or tricycle will overturn if the vertical through its C.G. falls outside the wheel base (Fig. 127).

Fig. 127.

(2) A porter carrying a heavy trunk in one hand often extends the opposite arm at full length in order to more readily bring his C.G. over a point between his two feet. A man carrying a heavy weight in front of him leans back in order to bring his C.G. over his base.

(3) If we lean too far back in a chair tilted up on its hind legs, we shall fall over backwards when the vertical through the C.G. falls behind the line joining the two back feet.

(4) The leaning tower of Pisa overlaps its base by 13½ feet at its highest point, but it stands because the vertical through its C.G. is well within the base. Treating the tower as a uniform cylinder (Fig. 126), its C.G. is at **G** midway between two diagonally opposite points **A**, **C** at the top and bottom. Hence the tower would fall if **G** were to project beyond **A**, that is, if the top were to overlap by more than the diameter of its base.

158. **Experimental verifications.**—The reader is strongly recommended to test the truth of the theorem by some simple experiments made with any common objects around him.

Thus, a book can be placed on the top of another book resting on a table, provided that the middle (or C.G.) of the upper book does not project beyond the lower book (*i.e.* beyond the *base*). Hence, in order to make the upper book project beyond its middle, weights must be placed on the supported end to bring the C.G. nearer that end.

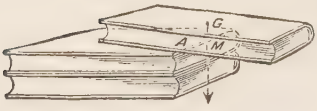


Fig. 128.

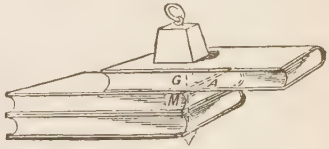


Fig. 129.

In like manner, if any body of rectangular section **ABCD** (say a brick) be stood upon a rough plank, and the plank be gradually tilted about the edge **A**, the body will remain standing as long as its C.G. does not overlap the base, but directly this happens it will fall over about its lowest edge (Figs. 125, 126).

Now the C.G. of a rectangular body lies in the diagonal plane **AC**. Hence the body will turn over just after the diagonal **AC** has passed through the vertical position.

159. **Stable, unstable, and neutral equilibrium.**—Theoretically, it is possible to balance a body by supporting it at a point either vertically *above* or vertically *below*, or *at* its C.G. But, practically, it is often very difficult to keep a body balanced on a point even for a short time, and the least disturbance suffices to overturn it. Bodies have, as we know, a tendency to fall into certain natural positions of equilibrium, and to fall away from other positions.

DEFINITION.—Bodies are said to be in **unstable** equilibrium when, after a slight disturbance, they tend to move further and further away from their equilibrium position, *i.e.* to upset.

Thus an egg naturally rests on a table with its side touching the table. But it is difficult to balance the egg on its end ; and, if this has been done by bringing the C.G. directly over the point of support, the egg will still overturn, as in Fig. 133, with the slightest shake or breath of air, or other disturbance, which moves the egg and therefore its C.G. or its point of support a little to one side or the other. In ordinary language, we express this fact by saying that the egg is *top heavy*.

A pin would theoretically satisfy the conditions of equilibrium of § 151 if stood upright with its point resting on a plate. But no hand is sufficiently steady and patient to place it exactly in the right position, nor could the plate and pin remain sufficiently undisturbed for the pin to continue balanced for more than an instant, even if it were so placed. The pin would in fact be *top heavy*.

A walking-stick standing upright on the finger, a ball or marble placed at the top of an inverted bowl, are also examples of unstable equilibrium.

NOTE.—*When we say that a body is "top heavy," we imply that it is in unstable equilibrium.*

DEFINITION.—Bodies are said to be in **stable** equilibrium when they tend to return to their equilibrium position after being slightly disturbed.

Thus, for example, a weight (such as a plummet) hanging from a string will of its own accord fall into a position of equilibrium with the string vertical. If pulled aside, it will at first swing to and fro, but the string will at last again assume a vertical position.

A stick hanging by its crook, a ball or marble inside a basin, the beam of a balance, are also examples of stable equilibrium.

DEFINITION.—Bodies are said to be in **neutral** equilibrium when, after being slightly displaced, they remain in their new position.

A ball whose C.G. is at its centre is in neutral equilibrium when placed on a horizontal surface. For, however the ball be rolled about, its C.G. will always be vertically above its point of contact, and hence it will always remain in equilibrium (Fig. 120, p. 198).

An egg or cylinder resting on its side when allowed to roll along the table, a heavy body of any kind supported *at* its C.G. (§ 160), a door turning on its hinge, are also examples of neutral equilibrium.

A *cone* affords a good illustration of all three kinds of equilibrium. On its base it is *stable*, on its vertex *unstable*, and on its side *neutral* for lateral displacements (rolling).

A top, a hoop, or a bicycle will remain upright as long as it is in rapid motion, although it would be unstable and would overturn if at rest. Similarly the Japanese in their balancing feats are able to keep bodies standing in positions of unstable equilibrium by moving the point of support slightly, so as to counteract every tendency to overturn. Hence, when a body is *in motion*, it is impossible to argue from purely statical considerations as to whether it is stable or not.

160. Stability of a body with one point fixed.

A heavy body, moveable freely about a fixed point **O**, is in stable, unstable, or neutral equilibrium, according as **O** is vertically above, vertically below, or at **G**, the body's C.G.

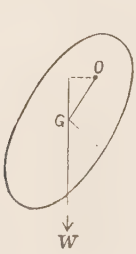


Fig. 130.

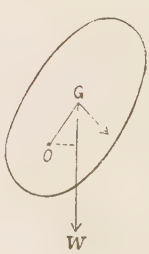


Fig. 131.

For if the line **OG** is not quite vertical, Figs. 130, 131 show that the moment of the weight acting at **G** tends to turn the body about **O** *towards* the position in which **G** is vertically below **O**, and *away* from the position in which **G** is vertically above **O**. Hence the former position is stable and the latter unstable. And since the body balances in every position when supported *at* **G**, its equilibrium is then neutral.

161. **Stability of a body resting on a horizontal plane.**—When a body rests touching the ground at one point, its equilibrium is not necessarily unstable, even although its C.G. is above its point of contact.

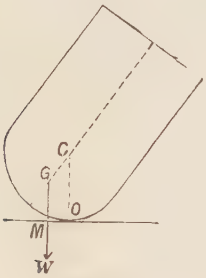


Fig. 132.

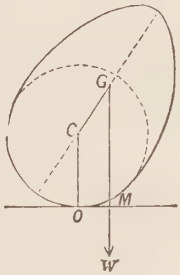


Fig. 133.

If the surface of a body is spherical and its C.G. is not at its centre, the body will be in stable equilibrium on a horizontal plane if its C.G. be vertically *below* its centre, and in unstable equilibrium if its C.G. be vertically *above* its centre; for Figs. 132, 133 show that the moment of the weight tends to turn the body *towards* the equilibrium position in the former case and *away* from it in the latter.

The whole body need not necessarily be spherical, provided that the part of the surface touching the plane is spherical.

162. Thus, if a hemisphere of lead is joined to a cylinder of cork so that the C.G. of the whole is below the centre of the hemisphere, it will stand upright although it *looks* very top-heavy. For lead is so much heavier than cork, that a cork of considerable length may be attached to the hemisphere without raising the C.G. above the centre (Fig. 132).

Several toys act on this principle, for, if the lead and cork be coated together with paint, the difference of materials is disguised, and so a doll which will only stand on its head is easily made.

163. **General condition of stability.**—The weight of a body acting at its C.G. always tends to pull this C.G. down towards the Earth. Hence, if the body is in equilibrium with the C.G. in the highest possible position, its equilibrium must necessarily be unstable. And when the C.G. is at the lowest possible position, the body must necessarily be in stable equilibrium, for gravity has pulled it as low as possible, and it can go no further.

For this reason, in order to make a body stable, it is often loaded

with weights low down, so as to lower the c.g. as much as possible. This is why ships carry ballast. On the other hand, people by standing up in a small boat raise the c.g., and *may* upset it.

164. As another illustration, a small particle or spherical ball will be in unstable equilibrium when resting at the highest point of the convex surface of a round body, and in stable equilibrium when resting at the lowest point of a concave bowl. The student should, as an exercise, draw diagrams for the two cases, showing that when the ball is displaced its weight tends to pull it away from the former but towards the latter position of equilibrium.

Examples.—(1) A square board **ABCD** has a mass of 10 lb. If it is suspended from the point **A**, what mass must be attached at **B**, in order that the board may rest with **AB** inclined to the horizontal at an angle of 60° ?

The resultant weight of the board acts at its centre **O**.

Let x lb. be the required mass, and a the length of each side of the square. Then each of the forces x lb. and 10 lb. tends to turn the board about **A**, and the board will be in equilibrium when the moments of these forces about **A** are equal and opposite ; *i.e.* when

$$x \times \mathbf{AE} = 10 \times \mathbf{AF}.$$

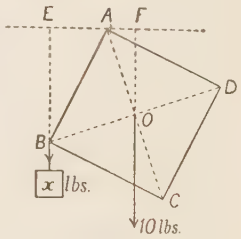


Fig. 133*a*.

Thus we have

$$x \times \mathbf{AB} \cos \mathbf{EAB} = 10 \times \mathbf{AO} \cos \mathbf{FAO},$$

or $$x \times a \cos 60° = 10 \times \frac{a}{\sqrt{2}} \cos 75°.$$

But

$$\cos 75° = \cos (45° + 30°) = \frac{\sqrt{3} - 1}{2\sqrt{2}} ;$$

whence $$x = \frac{10a \cos 75°}{\sqrt{2}a \cos 60°} = 5(\sqrt{3} - 1) \text{ or } 3{\cdot}66 \text{ lb.}$$

(2) In the accompanying figure, **AC** and **FD** are rectangles, and **H** is a point on the diagonal **EB**. Find the ratio **EH** : **EB**, if a piece of uniform cardboard in the shape of the figure **ABCDHFA** is on the point of toppling when it stands on the base **CD**.

This requires that the c.g. of the piece of cardboard should be vertically over the point **D** (§ 156). This gives the most convenient method of solving the problem, if we know how to find this centre of gravity (cf. § 173). But we can also proceed as follows :—

Let $$\mathbf{AB} = p, \ \mathbf{AE} = q, \ \mathbf{EH} = r \,.\, \mathbf{EB}.$$

Required to find r.

Now $\mathbf{ED} : \mathbf{EC} = \mathbf{EH} : \mathbf{EB}$,

hence $\mathbf{ED} = r \,.\, \mathbf{EC} = rp$.

Similarly $\mathbf{EF} = rq$.

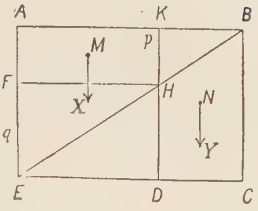


Fig. 133*b*.

Let us consider the weights of the rectangles **AH** and **KC** separately. Let **M** and **N** be their centres of gravity, X and Y their weights. Then if the body upsets it turns round the point **D**. Also X tends to upset the body, and Y to prevent it from upsetting. Thus the body will stand unless the moment of X about **D** exceeds the moment of Y about **D**, and it will be on the point of upsetting when these moments are equal.

i.e. $$X \times \tfrac{1}{2}\,\mathbf{FH} = Y \times \tfrac{1}{2}\,\mathbf{DC},$$

or $$X \times \tfrac{1}{2} rp = Y \times \tfrac{1}{2}(1 - r)\,p.$$

or $$\frac{X}{Y} = \frac{1 - r}{r} \qquad \text{(i.)}$$

Again, since the cardboard is uniform, the weights of these rectangles are proportional to their areas.

Thus $$\frac{X}{Y} = \frac{\text{area } \mathbf{AH}}{\text{area } \mathbf{KC}} = \frac{\mathbf{AF} \,.\, \mathbf{AK}}{\mathbf{KB} \,.\, \mathbf{BC}} = \frac{(1 - r)\,q \,.\, rp}{(1 - r)\,p \,.\, q} = r. \qquad \text{(ii.)}$$

From (i.) and (ii.) we have

$$\frac{1 - r}{r} = r\,; \quad \text{or} \quad r^2 + r - 1 = 0$$

whence $$r = \tfrac{1}{2}(-1 \pm \sqrt{5}) = \cdot 618,$$

(rejecting the negative root).

Summary of Results.

When a body is supported at one point, *its* C.G. *is in the same vertical line as the point of support.* (§ 151.)

The *base* of a body is the area enclosed by a string drawn tightly round the points of contact with the supporting surface. (§ 155.)

A body placed on a plane will *stand* or *fall* according as the vertical line through its C.G. falls *within* or *without its base.* (§ 156.)

A body is in *stable* or in *unstable equilibrium* according as when slightly displaced from its equilibrium position

it tends to *return to* or *move further away from* that position. (§ 159.)

A body is in *neutral equilibrium* if when slightly displaced it remains in its new position. (§ 159.)

EXAMPLES XI.

(*Selected Problems* :—3, 4, 5, 8, 9, 10, 12, 14, 15, 17, 18.)

1. Weights of 1 and 5 lb. are fixed at the two extremities of a uniform heavy bar 3 ft. long. The centre of gravity of the whole is 1 ft. from one end. Find the weight of the bar.

2. A rod, of which the centre of mass trisects its length, is hung from a smooth peg by means of a string attached to its extremities. Find the positions of equilibrium.

3. A rod 12 ft. long has a weight of 1 lb. suspended from one end, and when 15 lb. is suspended from the other end it balances at a point 3 ft. from that end, while if 8 lb. are suspended there it balances at a point 4 ft. from that end. Find the weight of the rod and the position of its centre of gravity.

4. A bar projects 6 in. beyond the edge of a table, and when 2 oz. is hung on to the projecting end the bar just topples over ; when it is pushed out so as to project 8 in. beyond the edge 1 oz. makes it topple over. Find the weight of the bar and the distance of its centre of gravity from the end.

5. A body of any form is supported by a string, which, passing over a smooth peg, is fastened at its extremities to two points in the body. Show that in its position of equilibrium the C.G. of the body is vertically beneath the peg, and that the two portions of the string make equal angles with the vertical direction.

6. Explain fully the circumstances under which a rectangular block, standing on a plank which is being gradually tilted, shall topple over, being prevented from sliding by a small obstacle. As an example, take the case of a block $8 \times 5 \times 5$ cubic in.

7. Masses of 1 lb., 1 lb., 2 lb., 2 lb. are placed at the corners **A, B, C, D** of a rectangle **ABCD**. Determine the centre of gravity of the four masses.

8. Of two equal hollow spheres of the same material, one contains a quantity of water, and the other a mass of lead attached to the inner surface. Describe experiments which would enable you to distinguish between them.

9. A number of rectangular bricks are placed so that each overlaps the one next below by the greatest amount possible. Show that the distances they overlap (beginning at the top) are proportional to the fractions, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{6}$, $\frac{1}{8}$, ...

10. A crane is on a four-wheeled railway truck weighing 6 tons. The area of the *base* formed by the wheels is 50 sq. ft. Neglecting the weight of the crane, calculate the area of the ground from which a mass of 3 tons can be lifted by the crane.

11. A rod whose length is 10 ft., and which is thicker at one end than at the other, balances about its centre when 10 lb. is hung from one end and 20 from the other; while if 40 lb., instead of 20, is hung from the second end the point about which it balances is at 4 ft. from that end. Find the weight of the rod and the position of its centre of gravity.

12. **ABCD** is a board in the shape of a parallelogram, standing with its side **BC** in contact with a rough inclined plane, **B** being on the lower level. If $BC = a$, $CD = b$, and the angle **C** is acute, prove that the board will not topple over unless the inclination of the plane exceeds $\tan^{-1}(a - b\cos C) / b \sin C$.

13. A rectangular board is resting in contact with the rough inner surface of a fixed circular hoop, both board and hoop being in a vertical plane. If the sides of the board are $2a$ and $2b$, (the side $2a$ forming the chord of the circle), show that the smallest possible depth of the centre of gravity of the board below the centre of the hoop is

$$b[\sqrt{(r^2 - a^2)} - b]/\sqrt{(a^2 + b^2)}.$$

14. A symmetrical three-legged stool has its centre of gravity at a distance h from the plane through the feet of its legs; and the feet form an equilateral triangle with sides of length $2a$. The stool rests

on a perfectly rough inclined plane with the two lowest feet in the same horizontal line. Prove that equilibrium is possible if the inclination of the plane to the horizon is less than $\tan^{-1}(a/h\sqrt{3})$.

15. A uniform rod **AB** of mass 1 lb. is 12 inches long, and is hinged at a point **C** 3 inches from **A**. Masses of 6 and 1 lb. are fastened at **A** and **B** respectively, and the rod is placed with **A** vertically over **C**. (i) Is equilibrium possible? (ii) If so, is it stable or unstable? (iii) What extra weight fastened at **B** would cause neutral equilibrium?

16. Prove that the pull exerted by a man in a tug of war is $\frac{a}{b}$-times his weight, when a is the horizontal overhang of his C.G. backwards over his heels, and b is the height of the rope above the ground.

17. A rectangular block of wood of weight 20 lb. and of breadth 1 foot stands on a horizontal plane; a rod of length 5 feet and of weight 12 lb. is to be nailed across the top. Find the greatest distance to which it may project over one edge of the block without upsetting it.

18. A light wire in the shape of an equilateral triangle **ABC** is suspended from the point **A**. If weights of 1 and 2 lb. respectively are fastened at **B** and **C**, show that in the position of equilibrium **BC** makes with the horizontal an angle $\tan^{-1}\frac{1}{5}\sqrt{3}$.

CHAPTER XII.

DETERMINATION OF THE CENTRE OF GRAVITY.

165. In the last chapter we showed how the position of the C.G. of certain figures—such as the straight line, sphere, and parallelogram—can be inferred from the symmetry of the figures. In most cases, however, it is necessary to divide a body into a number of parts, and to deduce the position of the C.G. of the whole body from those of its parts. Accordingly we commence by investigating the general method applicable in such cases.

166. Having given the weights and centres of gravity of the different parts of a body, to find the C.G. of the whole body.

Let S_1, S_2, S_3 be different parts of a body, and let w_1, w_2, w_3 be the weights, **K, L, M** the C.G.'s, of S_1, S_2, S_3, respectively. It is required to find the C.G. of the whole body made up of the parts S_2, S_3, S_1.

The weights w_1, w_2 may be taken to act at **K, L.** Therefore their resultant is a weight $w_1 + w_2$ acting at a point **O** on **KL**, such that

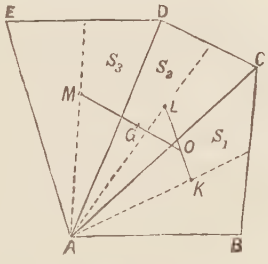


Fig. 134.

$$w_1 \,.\, \mathbf{KO} = w_2 \,.\, \mathbf{OL}.$$

$\therefore \quad (w_1 + w_2)\ \mathbf{KO} = w_2 \,.\, \mathbf{OL} + w_2 \,.\, \mathbf{KO} = w_2 \,.\, \mathbf{KL}$;

whence

$$\mathbf{KO} = \frac{w_2}{w_1 + w_2}\mathbf{KL}, \quad \mathbf{OL} = \mathbf{KL} - \mathbf{KO} = \frac{w_1}{w_1 + w_2}\mathbf{KL}.$$

Thus **O** is the C.G. of the body made up of S_1 and S_2.

The total weight of the two parts S_1 and S_2 is therefore $w_1 + w_2$ acting at **O**.

The weight of S_3 is w_3 acting at **M**. Therefore their resultant is a weight $w_1 + w_2 + w_3$ acting at a point **G**, such that

$$(w_1 + w_2)\ \mathbf{OG} = w_3 \,.\, \mathbf{GM}.$$

$\because \quad (w_1 + w_2 + w_3)\ \mathbf{OG} = w_3\ (\mathbf{OG} + \mathbf{GM}) = w_3 \,.\, \mathbf{OM}$,

whence

$$\mathbf{OG} = \frac{w_3}{w_1 + w_2 + w_3}\mathbf{OM}, \quad \mathbf{GM} = \frac{w_1 + w_2}{w_1 + w_2 + w_3}\mathbf{OM}.$$

Thus **G** is the C.G. of the body made up of S_1, S_2, S_3.

In like manner, if the weights and C.G.'s of four or more parts of a body are known, we can find the C.G. of the whole body. The required C.G. is evidently the C.G. of a number of particles whose weights are equal to those of the several parts of the body, placed at their respective C.G.'s, and the above method is identical with that used for finding the centre of parallel forces (p. 185).

Examples.—(1) To find the C.G. of a wire bent into an angle.

Let a uniform wire **AB** be bent into an angle **ACB**. We may consider it as consisting of two straight wires **AC**, **CB** joined together at **C**.

The C.G.'s of these two portions are at their middle points **K**, **L**, and their weights are proportional to their lengths **AC**, **CB**. Hence the C.G. of the whole wire is at a point **G** in **KL**, such that

$$\mathbf{AC} \times \mathbf{KG} = \mathbf{CB} \times \mathbf{GL},$$

or

$$\frac{\mathbf{GK}}{\mathbf{GL}} = \frac{\mathbf{CB}}{\mathbf{CA}}.$$

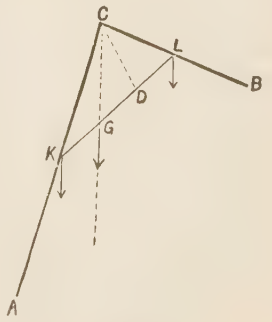


Fig. 135.

[The point G may be found by the following geometrical construction :—Draw CD bisecting the angle ACB and cutting KL in D. Take KG = DL, and therefore GL = KD. Then G is the required C.G.

For since CD bisects angle KCL, therefore (Euclid, VI. 3)

$$\frac{LD}{DK} = \frac{LC}{CK}; \quad \therefore \quad \frac{GK}{GL} = \frac{CL}{CK} = \frac{\frac{1}{2}CB}{\frac{1}{2}CA} = \frac{CB}{CA}. \ ]$$

(2) To find the C.G. of a cubical box without a lid.

Let a be the length of a side of the cube. Let O be the centre of the cube, A the centre of its base, G the required C.G. (figure should be drawn). Then it is easy to see that the C.G. of the four sides of the box is at O, and their total area is $4a^2$. Also the C.G. of the base is at A, and its area is a^2. Therefore G is the C.G. of weights a^2 at A and $4a^2$ at O.

$$\therefore \quad a^2 \times AG = 4a^2 \times GO.$$

$$\therefore \quad AG = 4GO, \quad \text{and} \quad AO = 5GO.$$

$$\therefore \quad GO = \tfrac{1}{5}AO, \quad \text{and} \quad AG = \tfrac{4}{5}AO.$$

But $AO = \frac{1}{2}a$. Therefore $AG = \frac{2}{5}a$.

Hence the C.G. is at a height above the base $= \frac{2}{5}$ the height of the cube.

167. DEFINITION.—The **medians** of a triangle are the straight lines joining its vertices to the middle points of its opposite sides.

In finding the C.G. of a triangle, it will be necessary to assume the following :—

LEMMA. *Any straight line, parallel to the base of a triangle and terminated by its sides, is bisected by the median through the vertex opposite the base.*

This is usually proved by the aid of similar triangles ; it can also be proved by Euclid, Book I., thus :—

Let ABC be the triangle, bc any line parallel to BC. Let D be the middle point of BC, and let AD cut bc in d. Then shall d be the middle point of bc.

For, if not, from dc, cut off dk = bd. Join Ak, and join D to b, c, k.

Then, since BD = DC ; $\therefore$ $\triangle$ ABD = $\triangle$ ACD and $\triangle$ bBD = $\triangle$ cCD,

$\therefore$ remaining $\triangle$ AbD = $\triangle$ AcD. And since bd = dk,

$\therefore$ $\triangle$ Abd = $\triangle$ Akd, and $\triangle$ Dbd = $\triangle$ Dkd ; $\therefore$ $\triangle$ AbD = $\triangle$ AkD ;

$\therefore$ $\triangle$ AcD = $\triangle$ AkD ;

$\therefore$ ck is parallel to AD, which is impossible, since ck cuts AD in d.

168. To find the C.G. of a triangular area.

Let **ABC** be a triangular lamina, **D, E, F** the middle points of the sides. Join **AD**. Then we shall show that the C.G. of the triangle lies in **AD**.

Suppose the triangle to be divided into a very large number of thin strips such as **bc**, by drawing straight lines parallel to the base **BC**.

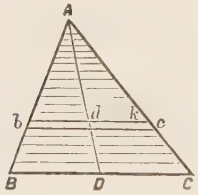


Fig. 136.

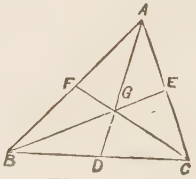


Fig. 137.

If the strips are made *sufficiently thin*, each may be treated as a uniform thin rod, and the C.G. of such a rod is at its middle point **d**.*

But, by the above lemma, the median **AD** passes through **d**. Hence the C.G. of every thin strip of the triangle lies in **AD**. And since the weight of each strip acts as if it were concentrated at its C.G., we see that the C.G. of the whole triangle is the same as that of a certain distribution of weights along the line **AD**.

Therefore the C.G. of the triangle lies in **AD**.

Similarly, by dividing the triangle up into strips parallel to **AC**, it may be shown that the C.G. of the triangle lies in the median line **BE**, and also in **CF**.

---

* We may suppose the triangle to be built up of a number of thin laths or strips of material, each slightly longer than the next above, and fixed side by side. Strictly speaking, their ends would have to be smoothed off along the sides **AB, AC**; but we suppose the strips so thin that the amount smoothed off is inappreciable.

Therefore the C.G. *of the triangle is at* **G**, *the common point of intersection of the medians* **AD, BE, CF.**

COR. *The three medians* **AD, BE, CF** *all pass through one common point.*

This is a well-known geometrical theorem.

169. The C.G. of a triangular area coincides with the C.G. of three equal particles placed at its vertices.

Also it is the point of trisection of any median line which is the more remote from the corresponding vertex.

(i.) Let three equal weights w be placed at **A, B, C.**

Then, if **D** be the middle point of **BC**, the weights w at **B** and w at **C** have a resultant $2w$ at **D.**

Hence the C.G. of the three weights is also the C.G. of weights $2w$ at **D** and w at **A.**

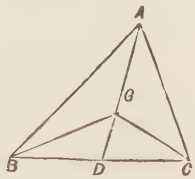


Fig. 138.

Therefore it lies in **AD.**

Similarly, it lies in the other two medians.

Therefore it is at **G**, their point of intersection.

That is, it coincides with the C.G. of the triangular area **ABC.**

(ii.) Again, since **G** is the point of application of the resultant of weights $2w$ at **D** and w at **A,**

therefore **G** divides **AD** so that

$$w \,.\, \mathbf{AG} = 2w \,.\, \mathbf{GD}.$$

$$\therefore \quad \mathbf{AG} = 2\mathbf{GD},$$

and

$$\therefore \quad \mathbf{AD} = \mathbf{AG} + \mathbf{GD} = 3\mathbf{GD}.$$

$$\therefore \quad \mathbf{GD} = \tfrac{1}{3}\mathbf{AD}, \quad \text{and} \quad \mathbf{AG} = \tfrac{2}{3}\mathbf{AD}.$$

Therefore **G** is that point of trisection of **AD** which is the more remote from the vertex **A.**

Cor. 1. If each of the weights w is *one-third* the weight of the lamina, their total weight and the position of their c.g. will be the same as for the lamina. Hence *a uniform triangular lamina is statically equivalent to three equal weights placed at its vertices, each being one-third the weight of the lamina.*

Cor. 2. *The point of intersection of the three medians of a triangle is one of the points of trisection of each of them.*

This may also be proved by pure geometry. The proof is left as an exercise for the reader.

170. **To find the C.G. of the perimeter of a triangle.**

Let **ABC** be a triangle. It is required to find the c.g. of a thin uniform wire bent into the triangle **ABC.**

Let **D, E, F** be the middle points of the sides **BC, CA, AB,** and let their lengths be a, b, c, respectively.

The weights of the three sides are proportional to their lengths, and act at their middle points. Hence the required c.g. is the c.g. of weights a, b, c placed at **D, E, F.**

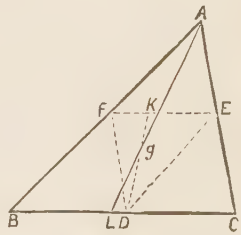


Fig. 139.

The weight a acting at **D** is equivalent to weights $\frac{1}{2}a$ at **B** and $\frac{1}{2}a$ at **C.** Similarly for each of the other sides. Therefore the c.g. of the perimeter is the c.g. of weights $\frac{1}{2}(b+c)$ at **A,** $\frac{1}{2}(c+a)$ at **B,** $\frac{1}{2}(a+b)$ at **C.**

It might, therefore, be found by § 134.

But the following method is more convenient : —

171. **Alternative construction.**—Since g is the c.g. of weights a, b, c placed at D, E, F, therefore Dg produced divides EF in a point K, such that $b \cdot EK = c \cdot KF$. But the sides of $\triangle$ DEF are, respectively, half those of ABC.

$$\therefore \quad \frac{EK}{KF} = \frac{c}{b} = \frac{2ED}{2DF} = \frac{ED}{DF};$$

whence (Euclid, VI. 3) DK bisects $\angle$ EDF. Similarly Eg, Fg bisect the angles FED, DFE.

Therefore g *is the centre of the circle inscribed in the triangle* DEF.

172. To find the C.G. of the area of any polygon.

Divide the polygon into triangles (S_1, S_2, S_3, Fig. 140) by joining one of its vertices **A** to the other vertices not already joined to it. Find **K, L, M,** the C.G.'s of these triangles. Then the required C.G. of the polygon is the C.G. of weights at **K, L, M** proportional to the areas S_1, S_2, S_3, and is therefore given by the construction of § 166.

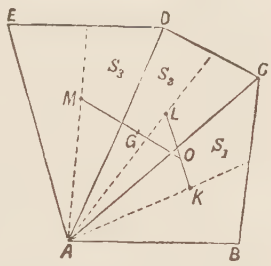


Fig. 140.

CAUTION.—The C.G. of a polygon is not, in general, the C.G. of *equal weights* placed at all the corners of the polygon.

173. To find the C.G. of a portion of a body.

Having given the weight and C.G. of a whole body and of any part removed from it, to find the position of the C.G. of the remaining part.

Let **O** be the C.G. of a body, W its weight.

Let **C** be the C.G. of any part of the body, w its weight.

Let **G** be the C.G. of the remainder of the body. It is required to find **G**.

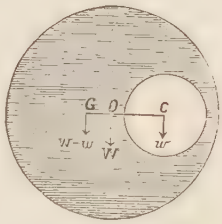


Fig. 141.

The weight of this remaining part is evidently $W - w$.

Now, since **O** is the C.G. of the whole body, **O** is the C.G. of weights w at **C** and $W - w$ at **G**.

$\therefore$ **C, O, G** lie in a straight line, and

$$(W - w)\ \mathbf{GO} = w\ .\ \mathbf{OC}.$$

Therefore **G** lies on **CO** produced through **O**, so that

$$\mathbf{OG} = \frac{w}{W - w}\ \mathbf{CO}.$$

Examples.—(1) **ABCD** is a square of paper, and **E, F** are the middle points of **AB, AD.**

(i.) To find the c.g. of the portion left when the triangle **AEF** is cut off.

(ii.) To find the c.g. of the whole when the triangle **AEF** is doubled over.

Let **O** be the centre of the square, and let $2a$ be the length of a diagonal so that $\mathbf{OA} = a$. Let W be the weight of the square; then that of $\triangle$ **AEF** is $\frac{1}{8}W$.

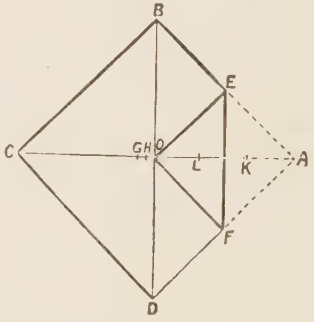


Fig. 142.

(i.) Now the c.g. of $\triangle$ **AEF** lies at **K** where $\mathbf{AK} = \frac{1}{3}\mathbf{AO}$, $\mathbf{OK} = \frac{2}{3}a$. Hence, if **G** be the c.g. of the remaining portion,

$$\tfrac{7}{8}W \times \mathbf{OG} = \tfrac{1}{8}W \times \mathbf{OK};$$

$$\therefore\ \mathbf{OG} = \tfrac{1}{7}\mathbf{OK} = \tfrac{2}{21}a = \tfrac{1}{21}\mathbf{AC}.$$

(ii.) When $\triangle$ **AEF** is folded over into the position **OEF** its c.g. is at **L** where $\mathbf{OL} = \frac{1}{3}\mathbf{a}$. Let **H** be the c.g. of the whole. Taking moments about **O**, we have to find the centre of $\frac{7}{8}W$ at **G** and $\frac{1}{8}W$ at **L**; hence, by § 137,

$$W \times \mathbf{OH} = \tfrac{7}{8}W \times \mathbf{OG} - \tfrac{1}{8}W \times \mathbf{OL};$$

$$\therefore\ \mathbf{OH} = \tfrac{7}{8}\mathbf{OG} - \tfrac{1}{8}\mathbf{OL} = \tfrac{1}{12}a - \tfrac{1}{24}a = \tfrac{1}{24}a = \tfrac{1}{48}\mathbf{AC}.$$

(2) To find the c.g. of a hollow spherical bullet containing an excentric spherical cavity.

Let **O** be the centre of the surface of the bullet, **C** the centre of the cavity (Fig. 141).

Let a, b be the known radii of the bullet and cavity, and let $\mathbf{OC} = c$. Then the volumes of the bullet and of the cavity are, respectively, $\frac{4}{3}\pi a^3$ and $\frac{4}{3}\pi b^3$.

Hence, if W denote the weight which the bullet would have if no cavity existed, w the weight of matter which would fill the cavity,

$$\frac{w}{W} = \frac{b^3}{a^3},$$

and the weight of the actual bullet is $W - w$.

Hence the required C.G. of the hollow bullet is a point **G** in **CO** produced, such that

$$\mathbf{OG} = \mathbf{CO} \times \frac{w}{W - w},$$

or

$$\mathbf{OG} = c \times \frac{b^3}{a^3 - b^3}.$$

(3) *To find the* C.G. *of a trapezoid* (*i.e.* a quadrilateral two of whose sides are parallel).

Let **ABCD** be the trapezoid, having **AC, BD** parallel. Let $\mathbf{AC} = a$, $\mathbf{BD} = b$, and let h be the perpendicular distance between **AC** and **BD**.

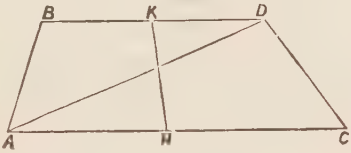


Fig. 143.

The quadrilateral may be divided into two triangles **ADC, ABD**, and the areas of these triangles are $\frac{1}{2}ha$, $\frac{1}{2}hb$; therefore their weights are proportional to a, b.

Their C.G. will therefore be unaltered by supposing their weights to be *equal* to a, b, respectively.

Then the C.G. and weight of the triangle **ADC** will be the same as that of equal weights $\frac{1}{3}a$ placed at **A, D, C**. Similarly, the triangle **ABD** may be replaced by equal weights $\frac{1}{3}b$ at **A, B, D.**

Hence the C.G. of the quadrilateral is the C.G. of the weights

$\frac{1}{3}(a + b)$ at **A**, $\frac{1}{3}a$ at **C**, $\frac{1}{3}b$ at **B**, and $\frac{1}{3}(a + b)$ at **D.**

Again, by dividing the quadrilateral into two triangles **ABC, BDC**, we see that its C.G. is also the C.G. of weights

$\frac{1}{3}a$ at **A**, $\frac{1}{3}(a + b)$ at **C**, $\frac{1}{3}(a + b)$ at **B**, and $\frac{1}{3}b$ at **D.**

Hence the C.G. will be the same when the latter weights are added to the former, and it is therefore the C.G. of weights

$\frac{1}{3}(2a + b)$ at **A**, $\frac{1}{3}(2a + b)$ at **C**, $\frac{1}{3}(a + 2b)$ at **B**, $\frac{1}{3}(a + 2b)$ at **D.**

Let **H, K** be the middle points of **AC, BD.**

Then the weights $\frac{1}{3}(2a + b)$ at **A** and $\frac{1}{3}(2a + b)$ at **C** are equivalent to a weight $\frac{2}{3}(2a + b)$ at **H.**

Similarly, the weights $\frac{1}{3}(a + 2b)$ at **B** and $\frac{1}{3}(a + 2b)$ at **D** are equivalent to a weight $\frac{2}{3}(a + 2b)$ at **K.**

Hence the C.G. of the quadrilateral is in **HK** at a point **G**, such that

$$\tfrac{2}{3}(2a + b) \times \mathbf{HG} = \tfrac{2}{3}(a + 2b) \times \mathbf{GK},$$

or

$$\frac{\mathbf{HG}}{\mathbf{GK}} = \frac{a + 2b}{2a + b}.$$

174. To find the C.G. of any number of weights at given points in one plane.

Let any number of particles of known weights w_1, w_2, w_3, . . be situated at given points $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$, . . in one plane. Draw two straight lines **OX**, **OY** at right angles to one another in the plane, and let the distances of each weight from each of these two lines be measured.*

Let x_1, x_2, x_3, . . be the distances of the weights from **OY**; y_1, y_2, y_3, . . . their distances from **OX**.

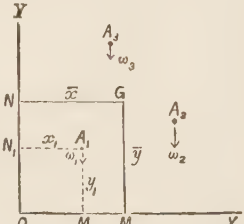


Fig. 144.

[So that if, from any weight $\mathbf{A}_1$, perpendiculars $\mathbf{A}_1\mathbf{M}_1$ on **OX** and $\mathbf{A}_1\mathbf{N}_1$ on **OY** be drawn, we have

$$x_1 = \mathbf{OM}_1 = \mathbf{N}_1\mathbf{A}_1 \text{ and } y_1 = \mathbf{ON}_1 = \mathbf{M}_1\mathbf{A}_1.]$$

Let **G** be the required C.G. of the weights. Draw **GM**, **GN** perpendicular on **OX**, **OY**, and let $\bar{x} = \mathbf{OM} = \mathbf{NG}$, $\bar{y} = \mathbf{ON} = \mathbf{MG}$.

The resultant of the weights w_1, w_2, w_3, etc., acting at $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$, is a weight of $w_1 + w_2 + w_3$. . acting at **G**.

Suppose the plane turned so that **OY** is vertical and **OX** horizontal. Then, since the sum of the moments of the several weights about **O** is equal to the moment of their resultant,

$$\therefore \quad \mathbf{OM} \times (w_1 + w_2 + w_3 + \ldots) = \mathbf{OM}_1 \times w_1 + \mathbf{OM}_2 \times w_2 + \mathbf{OM}_3 \times w_3 + \ldots$$

$$\text{or} \qquad \bar{x} = \mathbf{OM} = \frac{w_1x_1 + w_2x_2 + w_3x_3 + \ldots}{w_1 + w_2 + w_3 + \ldots}.$$

Next place the system so that **OX** is vertical and **OY** horizontal. By taking moments in like manner about **O**, we have

$$\mathbf{ON} \times (w_1 + w_2 + w_3 \ldots) = \mathbf{ON}_1 \times w_1 + \mathbf{ON}_2 \times w_2 + \mathbf{ON}_3 \times w_3 + \ldots$$

$$\text{whence} \qquad \bar{y} = \mathbf{ON} = \frac{w_1y_1 + w_2y_2 + w_3y_3 + \ldots}{w_1 + w_2 + w_3 + \ldots}.$$

Hence the distances **OM**, **ON** are known, and by completing the rectangle **OMGN**, the position of **G**, the required C.G., can be found.

* We may suppose the " particles in one plane " to be a number of small weights attached to a flat square sheet of cardboard, and the two straight lines at right angles to be two adjacent edges of the square.

OBSERVATIONS.—The formulae evidently apply to finding the C.G. of a body, when $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$ are the C.G.'s of its several portions, and w_1, w_2, w_3 their weights.

Again, if for w_1, w_2, w_3 we write m_1, m_2, m_3, the masses of the bodies, the formulae determine the position of their centre of mass.

Although the weights are supposed above to be in one plane, we may here state that similar formulae hold in the more general case. If x_1, y_1, z_1 are the distances of any weight $\mathbf{A}_1$ from three planes at right angles (say two adjacent walls and the floor of a room), and so on, the corresponding distances $\bar{x}$, $\bar{y}$, $\bar{z}$ for the C.G. are given by formulae proved above, and a third similar formula with z's written for x's or y's.

Example.—(1) To find the C.G. of a square slab **ABCD** whose weight is 1 lb., together with weights of 2, 3, 4, 5 lb., placed at its four corners.

Let x be the distance of the C.G. from **AD**, y its distance from **AB**, a the length of the side of the square.

The total weight = 1 + 2 + 3 + 4 + 5 lb. = 15 lb. Therefore, taking **AD** vertical, the equation of moments about **A** gives

$$15x = 2 \cdot 0 + 5 \cdot 0 + 1 \cdot \tfrac{1}{2}a + 3 \cdot a + 4 \cdot a = 7\tfrac{1}{2}a\,;$$
$$\therefore \quad x = \tfrac{1}{2}a.$$

Taking **AB** vertical, we have, in like manner,

$$15y = 2 \cdot 0 + 3 \cdot 0 + 1 \cdot \tfrac{1}{2}a + 4 \cdot a + 5 \cdot a = 9\tfrac{1}{2}a\,;$$
$$\therefore \quad y = \tfrac{19}{30}a.$$

Hence **G** lies on the bisector of the sides **AB**, **DC** at a distance $\frac{19}{30}a$ from **AB**.

175. Work done in raising weights.—*The work done in raising a number of weights off the ground or raising them up to the ground is the same as if their total weight were collected at their* C.G.

Let there be any number of weights w_1, w_2, w_3, . . , and let them be raised from the ground to heights x_1, x_2, x_3, . . . Let W be their total weight, $\bar{x}$ the height of their C.G. in the new position.

The work done in raising the weights

$$= w_1x_1 + w_2x_2 + w_3x_3 + \ldots$$

But, by the last article,

$$\bar{x} = \frac{w_1x_1 + w_2x_2 + w_3x_3 + \ldots}{w_1 + w_1 + w_3 + \ldots} = \frac{w_1x_1 + w_2x_2 + w_3x_3 + \ldots}{W}\,;$$
$$\therefore \quad w_1x_1 + w_2x_2 + \ldots = W \cdot \bar{x}\,;$$

$\therefore$ whole work done = $W\bar{x}$ = work required to lift total weight to the height of the C.G.

Similarly, by taking $x_1, x_2 \ldots$ to represent the depths of a number of weights *below* the ground, we see that the work done in lifting a number of weights from below the surface to the ground is the same as if the weights were all concentrated at their C.G.

OBSERVATION.—The weights need not all be in the same vertical plane. *The theorem is true in every case,* but, unless the weights are in a vertical plane, the above proof assumes the more general theorem which we stated without proof in the observation on § 174.

Examples.—(1) The work done in building a cylindrical tower is the same as would be required to lift the whole of the materials through $\frac{1}{2}$ the height of the tower.

(2) The work done in digging a ditch of triangular section through earth of uniform material is the same as would be required to lift the total mass of earth through $\frac{1}{3}$ the depth of the lowest point of the ditch.

176. To find the C.G. of a uniform *tetrahedron* or pyramid on a triangular base.—We shall now show* that—

(i.) *The* C.G. *of a triangular pyramid* **ABCD** *is in the line joining any corner* **D** *to* **H**, *the* C.G. *of the opposite face.*

(ii.) *It coincides with the* C.G. *of four equal weights placed at the corners* **A, B, C, D.**

(iii.) *It divides the straight line joining any corner to the* C.G. *of the opposite face in the ratio of* 3 *to* 1, *i.e. at a point* **G**, *such that* $DG = \frac{3}{4}DH$ *and* $GH = \frac{1}{4}DH$.

(i.) Let **F** be the middle point of **AB, H** the C.G. of the triangle **ABC.** It is required to show that the C.G. of the pyramid lies in **DH.**

Divide the pyramid into an infinitely large number of infinitely thin triangular laminae of uniform thickness by drawing planes parallel to the face **ABC,** and let **abc** be one of these sections.

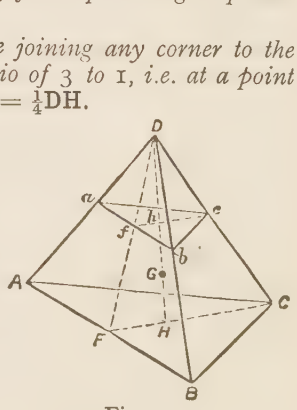


Fig. 145.

Join **DF,** cutting **ab** in **f,** and in the plane of the triangle **DCF** draw the straight lines **cf, DH** intersecting in **h.**

* The proofs may be omitted by the beginner, but the results stated are important.

Then, since **ab** is parallel to **AB** and **F** is the middle point of **AB**, therefore **f** is the middle point of **ab** (§ 167). Hence **ch** produced bisects **ab** in **f**; therefore **ch** is a median of the triangle **abc**. In like manner, **ah** and **bh** are medians of **abc**. Therefore **h** is the C.G. of the triangular lamina **abc**, and it lies in the line **DH**. Thus the C.G. of each of the laminae of the pyramid is in the line **DH**; therefore that of the pyramid is also in the line **DH**.

(ii.) Now let equal weights w be placed at the four corners **A, B, C, D**. The C.G. of the weights at **A, B, C** is **H**; hence that of the four weights also lies in the line **DH**.

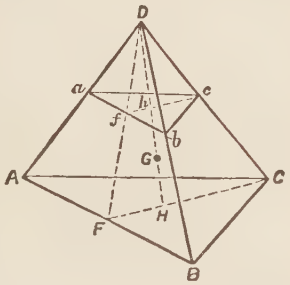


Fig. 145.

Similarly, if either of the other corners, such as **A**, be joined to **K**, the C.G. of the opposite face, both the C.G. of the pyramid and that of the four equal weights at **A, B, C, D** must lie in the joining line **AK**.

Therefore **AK** and **DH** must intersect in a point **G**, and **G** will be the C.G. both of the pyramid and of the four equal weights at **A, B, C, D**.

(iii.) Now the C.G. of the four weights w at **A, B, C, D** is the same as that of w at **D**, and $3w$ at **H** (since **H** is the C.G. of the weights at **A, B, C**). Therefore **G**, the C.G. of the pyramid, is a point in **DH**, such that

$$3\mathbf{GH} = \mathbf{DG}, \quad \text{or} \quad \mathbf{DG} : \mathbf{GH} = 3 : 1,$$

whence
$$\mathbf{GH} = \tfrac{1}{4}\mathbf{DH}, \quad \mathbf{DG} = \tfrac{3}{4}\mathbf{DH}.$$

COR. 1. **G** is the middle point of the line joining the middle points of opposite edges of the pyramid.

This follows at once by replacing the weights w at **A, B** by a single weight $2w$ at **F**, the middle point of **AB**, and the weights w at **C, D** by $2w$ at the middle point of **CD**.

COR. 2. The lines joining the four corners of a triangular pyramid to the C.G.'s of the opposite faces all pass through one common point **G**, and are there divided in the ratio of 3 to 1. Also the lines joining the middle points of pairs of opposite edges pass through and are bisected at the same point.

177. To find the C.G. of a pyramid on any polygonal base whatever.

Let **V** be the vertex, **ABCDE** the polygonal base of the pyramid. Let **H** be the C.G. of the base. Then shall the C.G. of the pyramid be a point **G** on **VH**, such that

$\mathbf{VG} = 3\mathbf{GH}$, or $\mathbf{GH} = \frac{1}{4}\mathbf{VH}$;

and $\therefore \quad \mathbf{VG} = \frac{3}{4}\mathbf{VH}$.

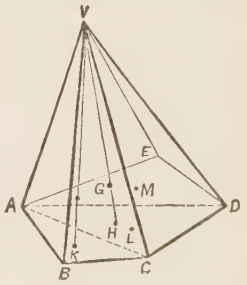


Fig. 146.

For divide the pyramid into a number of triangular pyramids, having **V** for vertex and the triangles **ABC, ACD, ADE** for bases. Let **K, L, M** be the C.G.'s of these bases.

Consider the triangular pyramid **VABC**. Its C.G. divides **VK** in the proportion of 3 : 1; therefore it may be replaced by $\frac{1}{4}$ its weight at **V** and $\frac{3}{4}$ its weight at **K**.

Let the weights of the other triangular pyramids be similarly replaced. Then the weights thus placed at **K, L, M** are proportional to the volumes of the pyramids, and therefore to the areas of their bases (since they have the same altitude). Hence the C.G. of these weights is **H**, the C.G. of the area of the base.

Therefore the whole pyramid is replaced by $\frac{1}{4}$ its total weight at **V** and $\frac{3}{4}$ its total weight at **H**. Therefore its C.G. divides **VH** so that $\mathbf{VG} = 3\mathbf{GH}$, as was to be proved.

178. To find the C.G. of a cone.

Draw any polygon circumscribing the base of the cone and complete the pyramid, having this polygon for base, and having its vertex at the vertex of the cone. This pyramid will circumscribe the cone, but, if the number of faces of the pyramid be made sufficiently great, the pyramid will not differ perceptibly from the cone.*

* For fresh faces may be added by slicing off the sharp edges of the slant surface of the pyramid, and this process continued till the pyramid is smoothed down to a cone.

Hence what holds good for the C.G. of a pyramid must also hold good for the C.G. of a cone.

Therefore the C.G. of a cone is in the line joining the vertex to the C.G. of the base, and at a distance from the latter point equal to $\frac{1}{4}$ the distance of the vertex.

In a *right circular cone* (the only kind of cone we have to consider) the C.G. is, of course, in the *axis*, at a distance from the base of $\frac{1}{4}$ the altitude of the cone.

179. **To find the C.G. of the *slant surface* of a regular pyramid, and of a right circular cone.**

The slant surface of the pyramid whose vertex is **V** and base any polygon **ABCDE**.. consists of a number of triangles **VAB, VBC, VCD**. The C.G. of each triangle is at a distance from its base equal to $\frac{1}{3}$ the distance of the vertex **V**.

Hence the C.G. of the whole surface is at a height above the base equal to $\frac{1}{3}$ the altitude of the pyramid.

And therefore, if the pyramid is *regular*, the C.G. must lie in its axis **VH** (which is perpendicular to and passes through **H** the centre of the base) at a point **G**, such that

$$\mathbf{HG} = \tfrac{1}{3}\mathbf{HV} \quad \text{and} \quad \mathbf{GV} = \tfrac{2}{3}\mathbf{HV}.$$

If the number of faces of the pyramid be sufficiently increased, the slant surface of the pyramid will ultimately become the slant surface of a cone.

Hence the C.G. of the slant surface of a right circular cone is on its axis, at a distance from the base equal to $\frac{1}{3}$ of the altitude.

OBSERVATIONS.—In the above investigation, the surface of the *base* is not taken into account. If we wanted the C.G. of the whole surface-area, we should have to apply § 166 or § 174.

The C.G. of the slant surface of *any* pyramid or cone whatever is at a distance from the base equal to $\frac{1}{3}$ the altitude; but, unless the pyramid or cone is symmetrical about its axis, there is no simple rule for finding the point where the line **VG** produced meets the base. This point is *not*, in general, the C.G. of either the area or perimeter of the base.

SUMMARY OF RESULTS.

The centre of gravity of

(1) *a triangular area* is the point of trisection of any median furthest from the corresponding vertex. It coincides with the C.G. of three equal particles placed at its vertices. (§§ 168, 169.)

(2) *a pyramid* or *a right cone* is in the line joining its vertex to the C.G. of the base at a distance from the base equal to $\frac{1}{4}$ of the distance of the vertex from the base. (§§ 176-178.)

If W, w be the weights of a body and of a portion of it, and if their C.G.'s are at **O** and **C**, the C.G. of the remaining portion is at a point **G** in **CO** produced through **O**, such that

$$\mathbf{OG} = \frac{w}{W - w}\,\mathbf{CO}. \qquad (\S\ 173.)$$

If $x_1, x_2, x_3 \ldots, y_1, y_2, y_3 \ldots$ be the distances of a series of weights $w_1, w_2, w_3 \ldots$ from two perpendicular straight lines **OY, OX**, respectively, the distances $\bar{x}, \bar{y}$ of the C.G. of these weights from **OY, OX** are given by

$$\bar{x} = \frac{w_1x_1 + w_2x_2 + \ldots}{w_1 + w_2 + \ldots}, \quad \bar{y} = \frac{w_1y_1 + w_2y_2 + \ldots}{w_1 + w_2 + \ldots} \qquad (\S\ 174.)$$

The *work done in raising a number of weights* is the same as if their total weight were collected at their C.G. (§ 175.)

EXAMPLES XII.

(*Selected Problems* :—2, 4, 5, 7, 10, 12, 13, 16, 18, 19, 22, 24, 26, 28, 29, 30, 33, 34, 38, 39, 40.)

1. How would you determine the centre of gravity of a hoop that was not quite circular?

2. A uniform isosceles triangle has its two equal sides each 5 ft. long, and its base 8 ft. long; find its centre of gravity. If its weight be 5 lb., and a weight of 10 lb. be hung at the vertex, find the centre of gravity of the whole.

3. Weights of 2 lb., 7 lb., 9 lb., 4 lb. are placed at the corners **A, B, C, D**, respectively, of a square, the length of whose side is 1 ft. Find the position of the C.G. of the weights.

4. Weights of 3 lb., 5 lb., 2 lb., and 6 lb. are placed at the corners **A, B, C, D**, respectively, of a square, the length of whose side is 8 in. Find the position of the centre of gravity of the weights.

5. **ABCD** is a square, **O** the point of intersection of its diagonals, **E, F** the middle points of the sides **AB, AD**. If the square **AEOF** be removed, find the centre of gravity of the remainder.

6. Find the centre of gravity of the remaining portion of a parallelogram when a triangle has been cut off from the parallelogram by a single straight cut.

7. **ABCDEF** is a regular hexagon, having **O** as the centre and a as radius of its inscribed circle. If the triangle **AOB** be removed, find the centre of gravity of the remaining area.

8. Assuming the rule for finding the centre of gravity of a triangular pyramid, prove the rule for finding that of a pyramid whose base is a four-sided figure.

9. Prove that if equal triangles be cut from the corners of a given triangle by lines parallel to the respective opposite sides the centre of mass of the remainder will coincide with that of the triangle.

10. If there are two triangles on the same base and between the same parallels, prove that the distance between their centres of gravity is one-third of the distance between their vertices.

11. **ABCD** is a square, **O** the intersection of its diagonals. If the triangle **AOB** is removed, find the centre of gravity of the remainder.

12. If the angular points of one triangle lie at the middle points of the sides of another, show that the centres of gravity of the triangles are coincident.

13. A cylinder of metal is 1 ft. high, 1 ft. external diameter and 11 in. internal diameter and 11 in. deep inside. It is open at the top. Find the position of its centre of gravity.

14. Determine the C.G. of a cone and of a frustum of a cone.

15. A wire 1 ft. long is bent into a right-angled triangle whose sides are 3, 4, 5 in. long. Apply the method of § 174 to find the distances of its C.G. from the sides including the right angle.

16. Find the number of foot-pounds of work required to wind up a given chain which hangs by one end.

17. Find the C.G. of a circular board, from which a circular piece has been cut out, having as diameter a radius of the board.

18. Prove that a triangular plate cannot stand vertically with its base resting on a horizontal plane if its vertex overhangs the base by more than the length of the base.

19. Three particles are situate at **A, B, C.** Prove that, by properly adjusting the ratios of their masses, their centre of gravity can be made to occupy any given position within the triangle **ABC.**

20. Find the ratios of the masses in Question 19 when the centre of gravity is at the centre of the inscribed circle.

21. A circular wire is divided into two parts by a chord. Prove that the distances of the centres of gravity of the parts from the centre of the circle are inversely proportional to the lengths of the parts.

22. A solid figure is formed of an upright triangular prism surmounted by a pyramid ; if the length of every edge of this figure be a ft., find the height of its centre of gravity above the base.

23. A square uniform plate is suspended at one of its vertices, and a weight equal to half that of the plate is suspended from an adjacent vertex of the square. Find the point where the vertical through the point of suspension cuts the opposite diagonal of the square.

24. Two sides of a rectangle are double of the other two, and on the longer side an equilateral triangle is described. Find the centre of gravity of the lamina made up of the rectangle and the triangle.

25. Masses of 1, 2, 3, 4, 5, 6, 7, 8 lb. respectively are placed at the corners of a cube **ABCDEFGH,** whose edge is 1 ft. and whose faces **ABCD, EFGH** are horizontal, **ABCD** being uppermost. How many ft.-lb. of work are done in exchanging the masses at **A, B, C, D** with those at **E, F, G, H** ? Hence find the vertical distance through which the common centre of gravity has been raised.

26. A shaft, 560 ft. deep and 5 ft. in diameter, is full of water ; how many ft.-lb. of work are required to empty it ?

27. **ABC** is a plane triangle. Weights of 2 lb., 2 lb., and 1 lb. are placed at the vertices, and their centre of gravity **G** is found. Then weights of 8 oz., 8 oz., and 14 oz. are placed at the same vertices, and their centre of gravity **H** is found. Prove that **G** and **H** are equally distant from the centre of gravity of the triangle.

28. Two circles of radii a, b touch one another internally, and the space between them is cut out of paper. Show that the distance of its C.G. from the point of contact of the circles is $(a + ab + b) \div (a + b)$, and find this distance when b approaches and becomes equal to a.

29. How would you place masses 1, 1, and $\sqrt{2}$, at the circumference of a circle, so that their centre of gravity should fall at the centre?

30. On the base **AB** of an equilateral triangle **ABC**, on the side remote from **C**, is described a rectangle whose height is half **AB**. Prove that the centre of gravity of the complete figure is the middle point of **AB**.

31. **AB** is a straight piece of wire 5 in. long, and on one side of it there are attached, at right angles, straight pieces of similar wire 1, 2, 3, 4, and 5 in. long respectively, the length of each piece being equal to the distance from **A** of its point of attachment, and all the pieces being in one plane. From what point in **AB** must the system be suspended in order that **AB** should rest at an angle of 45° with the vertical?

32. A tricycle weighing 4 st. 4 lb. has a small wheel symmetrically placed 3 ft. in front of two large wheels, which are 3 ft. apart. If the centre of gravity of the machine be 12 in. in front of the axle of the large wheels, and that of the rider, who weighs 10 st., be 9 in. in front of the axle, find the pressure of the different wheels on the ground.

33. A uniform lamina in the shape of a right-angled triangle, such that one of the sides containing the right angle is three times the other, is suspended by a string attached to the right angle. Prove that in the position of equilibrium the hypotenuse makes with the vertical an angle $\sin^{-1}\frac{3}{5}$.

34. If a given weight be placed anywhere on a given heavy uniform triangular lamina, the centre of gravity of the whole will lie within a certain triangle.

35. In a quadrilateral **ABCD** the sides **AB, AD** are each 15 in. and **BC, CD** each 20 in. in length. If **BD** = 24 in. find the distance of the centre of gravity of the quadrilateral from **A** and **C**.

36. A square lamina **ABCD** is suspended from **A**; weights of 4, 5, and 16 lb. are attached to it at **B, C,** and **D,** respectively; the vertical through **A** bisects the side **CD**. Find the weight of the square.

37. **ABC** is a triangular lamina, and **D, E** are the middle points of **CA, AB**. **ADE** is folded over **DE** till it lies flat upon the rest of the triangle. Show that the distance of the centre of gravity of the folded lamina from **BC** is $\frac{3}{4}$ of the distance of the centre of gravity of the unfolded lamina from **BC**.

38. A triangle **ABC**, of uniform wire, has the side **AC** removed, and is hung up by the point **A**. Show that, for **BC** to be horizontal

$$b^2(2a + c) = c(a + c)^2.$$

39. A piece of uniform cardboard in the form of a quadrilateral **ABCD** can just stand on the base **AB** without toppling over. If the angles at **A** and **D** are right angles, and if **AB**, **AD** measure 4 and 3 inches respectively, find the length of **DC** correct to one decimal place.

40. A piece of uniform cardboard in the shape of an isosceles triangle of altitude h and base $2r$ is standing on its base on a rough horizontal plane. To what vertical height can we raise one end of the base without upsetting the triangle?

EXAMINATION PAPER V.

1. The centre of gravity of a quadrilateral lies in one of the diagonals. Prove that one of the diagonals is bisected by the other.

2. How would you test the nature of the equilibrium of a body at rest? Point out the advantages of three-legged and four-legged tables respectively.

3. Find the centre of six like parallel forces of 3, 2, 3, 4, 5, 2 lb. acting at points **A, B, C, D, E, F** in a straight line, if **AB, BC, CD, DE, EF** = 2, 2, 4, 4, 3 in. respectively. Where would the resultant act if the forces at **D** and **E** were in the opposite direction to the rest?

4. The lid of a cubical box of uniform thickness is turned back till it comes into a horizontal position. If each edge of the box measures 12 in., what is the position of the centre of gravity of the box?

5. Find the centre of gravity of a uniform triangular pyramid; and prove that its position coincides with that of four spheres of equal weight centred at the four angles of the pyramid.

6. The triangle formed by joining the middle points of the sides of a given triangle is removed. Find the C.G. of the remainder.

7. A piece of uniform wire is bent into the shape of an isosceles triangle whose sides are respectively a, a, b. Show that the distance of the C.G. from the base of the triangle is $a\sqrt{a^2 - \frac{1}{4}b^2}/(2a + b)$.

8. A piece of uniform board in the shape of a triangle **ABC** is standing on a table on the base **AC**. If the sides **AB, BC, CA** are of lengths 3, 4, 5 inches respectively, how far may we push the corner **C** beyond the edge of the table without upsetting the board?

9. If two sides **AB, AC** of a triangle **ABC** are formed of uniform wire, and if it be suspended from **A**, show that the inclination of **AB** to the vertical is $\tan^{-1} b^2 \sin \mathbf{A} / (b^2 \cos \mathbf{A} + c^2)$.

CHAPTER XIII.

BALANCES.

180. In this chapter we shall describe the various contrivances by which bodies are usually **weighed**. Remembering that weight is proportional to mass, we observe that the operation of weighing by balancing a body with known weights affords in every case a correct measure of the *mass* or quantity of matter in the body in pounds or grammes or other chosen units, and that the observed *weight* is independent of any local variations in the intensity of gravity.

181. The common balance (see Fig. 147, p. 234) consists essentially of a **beam** or lever **AB** fixed so that it can turn about a fulcrum **O** placed a little above its middle point. From its ends are suspended two scale pans; the goods to be weighed are placed in one of these, and are balanced by placing suitable weights in the other, till the beam assumes a horizontal position.

In delicately constructed balances, the fulcrum and points of suspension consist of wedge-shaped pieces of hard steel (called "**knife blades**"), whose edges rest on hard plates of steel.

The requisites of a good balance are that it be

(i.) **true,** (ii.) **stable,** (iii.) **sensitive,** (iv.) **rigid.**

182. **Conditions that the balance may be true.**—A balance is said to be **true** if the beam assumes a horizontal position when equal weights are placed in the two scale pans. This requires that—

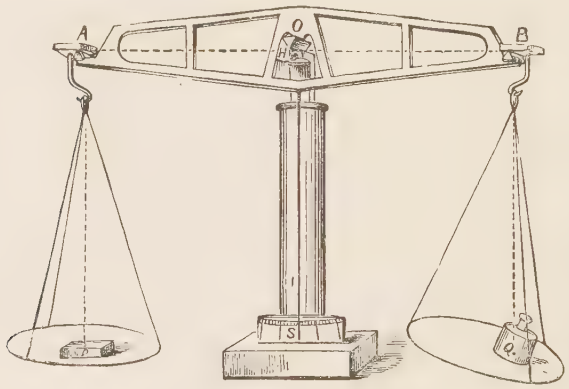


Fig. 147.

(i.) *The two arms of the beam must be of equal length,* that is, **AO** = **BO**, or the fulcrum **O** must be in a line **HO**, bisecting at right angles the line **AB**, which joins the points of suspension of the two scale pans.

(ii.) *The scale pans must be of equal weight.*

(iii.) *The* C.G. *of the beam must be vertically under the fulcrum when the beam is horizontal,* and therefore also in **HO**.

When these conditions are satisfied, equal weights placed *anywhere whatever* in the scale pans will balance each other with the beam horizontal.

For since the scale pans hang freely from the beam at **A** and **B**, they will assume positions in which the C.G. of each pan and its con-

tents is vertically below its points of suspension.* Hence the total weights of the pans and their contents always act vertically through **A, B** (Fig. 148, p. 236). These weights are equal; therefore their resultant acts at **H**, the middle point of **AB**. Also the weight of the beam acts at **G**, its C.G. But when the beam is horizontal, **G, H** are both vertically below the fulcrum **O**. Hence the resultant forces at **G, H** have no moment about **O**, and the beam is in equilibrium.

183. **Conditions that the balance may be stable.**—A balance is said to be **stable** if the beam tends of its own accord to fall into its equilibrium position. A balance would evidently be useless for weighing if its equilibrium position were *unstable* or even *neutral* (§ 159).

A balance is said to be more or less stable according to the comparative readiness or reluctance of the beam to assume its equilibrium position.

Stability is secured by placing **O**, the fulcrum of the beam, a little *above* the points, **G, H**, at which the resultant weights of the beam and the two pans act respectively.

For, if the beam be slightly inclined (Fig. 148), the equal weights of the two loaded scale pans still act at **A, B**, and their resultant therefore still acts at **H**†. And the moments about **O** of this resultant at **H**, and the weight of the beam at **G**, both tend to restore equilibrium by bringing the beam back to its horizontal position.

If the fulcrum **O** is only at a *very little* height above **G, H**, the moment tending to restore equilibrium will be very small, and the balance will oscillate for a long time before coming to rest, and will therefore possess very little stability. If goods have to be weighed quickly, the stability may be increased by increasing the height of the fulcrum **O**, thereby increasing the moment about **O** for any inclination of the beam. This, however, diminishes the sensitiveness of the balance, as we shall now explain.

---

* If one of the weights (*Q*, Fig. 147) is placed a little on one side in the scale pan instead of in the middle, the pan will swing itself a little to the other side so as to bring its C.G. vertically below the point of support **B**. The student should verify this by experiment.

† The point **H** is not the C.G. of the scale pans and their contents, but is the centre of parallel forces for their weights acting at **A, B**. It may or may not coincide with **G**.

184. **Conditions that the balance may be sensitive.**—It is not sufficient that the beam should be horizontal when the weights in the scale pans are equal. It must also indicate when they are *unequal*, by the beam assuming a non-horizontal position. This is expressed by saying that the balance must be **sensitive** (or, as some writers call it, "sensible"). In a sensitive balance, even a small additional weight placed in one scale pan should turn the beam through a perceptible angle, and the smallness of the weight which suffices to do so affords a measure of the sensitiveness of the balance and of the degree of accuracy attainable in weighing with it.

Thus a good chemical balance will indicate differences of weight down to tenths of a milligramme.

In order to enable the smallest deflection to be observed with great accuracy, the beam carries an index or pointer **I** (Fig. 147), which moves in front of a fixed graduated scale **S**, always remaining perpendicular to **AB**.

Sensitiveness may be secured at the expense of stability by making **OH**, the height of the fulcrum, small, and also by lengthening the arms.

For let a be the length of the arms **AH, HB**, Q the weight of each scale-pan and its contents, W that of the beam.

Let a small additional weight w be placed in the scale-pan at **A**. The moment of this weight about **O** is initially wa, and it turns the beam out of the horizontal position until it is balanced by the moment of the original weights (*i.e.* W at **G** and $2Q$ at **H**), tending in the opposite direction. Hence the greater the sensitiveness, the smaller must be the moment tending to restore equilibrium when w is removed, and the smaller therefore the stability of the balance.

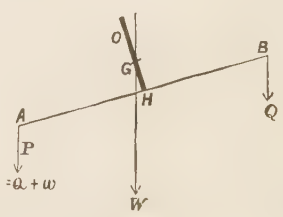


Fig. 148.

By increasing the arm a, we may obtain the same moment with a smaller weight w, and thus increase the sensitiveness, whilst, *according to the last article*, the stability would be unaltered, since it depends only on the heights **GO, HO**. *Practically*, however, the effect of lengthening the arms is to make the beam oscillate for a longer

time before coming to rest; hence it is impossible to attain rapid weighing with a highly sensitive chemical balance.

In a balance so constructed that **G, H** coincided with the fulcrum **O**, the beam would be in neutral equilibrium in any position with equal weights in the scale pans, but the slightest inequality in the weights would turn the beam right over. Such a balance would be highly sensitive, but would have no stability, and would therefore be practically useless.

185. **Rigidity.**—The balance must have a beam sufficiently strong not to bend under the weights which it has to carry. For this purpose a short thick beam would be preferable to a long thin one, but it would of course be less sensitive. To secure the greatest strength consistent with lightness, the beam is usually made in the form shown in Fig. 147.

186. **False balances.—Double weighing.**—A balance will evidently be false if—

(i.) The arms are of unequal length.

(ii.) The scale pans are of unequal weight.

(iii.) The beam is improperly balanced (*i.e.* **G** not on **OH**).

In all such cases the true weight of a body may be found by either of the following *methods of double weighing*.

The first method is to place the body in one scale pan and balance it with suitable counterpoises (*e.g.* small shot or fine sand) placed in the opposite pan. Now remove the body and replace it by weights sufficient to balance the counterpoises, and to bring the beam to the same position as before. These weights are evidently equal to the required weight of the body, however false the balance used, for they act under exactly the same circumstances and produce exactly the same effect.

The second method is less simple, but it enables us to test the trueness of the balance. The body is weighed first in one scale pan and then in the other. If the two observed weights are equal, the balance is true, and each is equal to the true weight of the body. If not, the balance is false, and we have the following cases to consider :—

187. Case 1. **When the arms are of unequal length,** but the weights of the scale pans balance one another with the beam horizontal.

Let a, b be the lengths of the arms. Let W be the true weight of a body, P, Q the weights required to balance it when it is weighed first in one scale pan and then in the other. Then, by taking moments about the fulcrum, we have $W \times a = P \times b$ .....(i.),

and $W \times b = Q \times a$ .....(ii.).

By multiplication, $W^2ab = PQab$, or

$$W^2 = PQ.$$

Therefore $W = \sqrt{(PQ)}$ .....(iii.),

that is, *the true weight is the geometric mean* between the observed weights.*

The moments of the weights of the beam and scale pan do not come into the above equations, because they balance one another in the horizontal position of the beam.

When the inequality of the arms is small (as is the case in all actual balances) it is sufficiently accurate for all practical purposes to take as the true weight the arithmetic mean $\frac{1}{2}(P + Q)$ instead of the geometric mean $\sqrt{(PQ)}$, which would be harder to calculate.

To compare the lengths of the two arms, we have, by (i.), (ii.),

$$\frac{a}{b} = \frac{P}{W} \quad \text{and} \quad \frac{a}{b} = \frac{W}{Q}$$

therefore $$\frac{a^2}{b^2} = \frac{WP}{WQ} = \frac{P}{Q}, \quad \text{or} \quad \frac{a}{b} = \sqrt{\left(\frac{P}{Q}\right)},$$

giving the ratio a/b. When this is known, the true weight of any body may be found by weighing once and multiplying the observed weight by a/b or b/a, according to which arm the body is suspended from.

Examples.—(1) The arms of a balance are in the proportion of 9 : 10. Sugar is weighed out against $\frac{1}{2}$-lb. weight, placed first in one scale pan and then in the other. To find the total true weight of sugar.

Since the two portions of sugar balance the $\frac{1}{2}$-lb. weight in the two pans, their actual weights are $\frac{1}{2} \times \frac{9}{10}$ and $\frac{1}{2} \times \frac{10}{9}$ lb.; therefore true weight of sugar $= \frac{1}{2}\left(\frac{9}{10} + \frac{10}{9}\right) = \frac{181}{180} = 1\frac{1}{180}$ lb.

(2) The observed weights of a body when weighed first in one scale pan and then in the other are 101 lb. and 99 lb. If the discrepancy

* $\sqrt{(PQ)}$ is the *geometric mean* between P and Q.

is due to the unequal length of the arms of the balance, find the error which would be made in taking the true weight to be 100 lb.

The true weight $= \sqrt{(99 \times 101)} = \sqrt{9999}$ lb.

By calculation, $\sqrt{9999} = 99{\cdot}995 = 100 - {\cdot}005$, approximately.

Hence, by taking the arithmetic mean of the two weights, viz. 100 lb., as the true weight, we should only make an error of ·005 lb., or $\frac{1}{2000}$ per cent. of the whole, whereas, if we were to take either of the observed weights 101 or 99 lb., we should make an error of about 1 lb., or 1 per cent.

188. Case 2. **When the scale pans are of unequal weight,** but the arms are of equal length.

This often happens in old balances on account of the greater wear and tear of the pan with the larger surface. It is sometimes corrected by fastening a piece of lead below the lighter pan, or filing down the heavier pan. If this is not done, the beam will not be horizontal when the pans are empty, so that the defect will be obvious.

Let p, q be the weights of the two pans, P, Q the weights which, when placed in them, respectively, will balance a body whose true weight is W.

Then, since the total weights on the two sides of the beam are equal,

$$\therefore \quad P + p = W + q,$$
$$W + p = Q + q;$$
$$\therefore \quad P - W = W - Q \quad \text{or} \quad 2W = P + Q;$$
$$\therefore \quad W = \tfrac{1}{2}(P + Q);$$

that is, *the true weight is the arithmetic mean of the observed weights.*

189. Case 3. **When the C.G. of the beam is a little on one side,** this also will throw the beam slightly out of the horizontal, and the effect will be the same as that produced by a slight inequality in the weights of the pans, from which, however, it may be distinguished by interchanging the two pans. It is corrected by filing the beam away on the heavier side till it balances horizontally. As in Case 2, the arithmetic mean of the two weighings gives the correct weight.

It follows that *the correct weight of a body is always the arithmetic mean of its apparent weights in the two scale pans except when the inequality in the arms of the balance is considerable.* [But even in this case the *first* method of double weighing of § 186 is correct.]

190. **Roberval's Balance.**—This is a letter-balance in which the scale pans are hinged to two levers **AEB, CFD,** which turn about their middle points **E, F.** When the balance is slightly displaced, one of the platforms goes up and the other goes down through exactly the same distance, and the platforms always continue to remain horizontal. Hence, if equal weights P, Q are placed anywhere on the pans, the works done by them in rising and falling will be equal and opposite, and will be the same, no matter whereabouts on the pans P, Q be placed. Therefore, by the Principle of Work, equal weights will balance one another whatever be their positions, although one may be nearer the fulcrums **E, F** than the other.

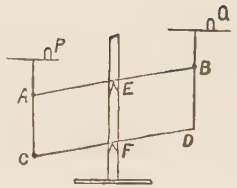


Fig. 149.

191. The common or Roman steelyard consists of a beam (**AB,** Fig. 150) moveable about a fulcrum or knife blade **C** fixed near one end **B.** From **B** is suspended the scale pan containing the body to be weighed, and a moveable weight is slid along the arm **CA** until the beam balances horizontally. The arm is graduated in such a way that the reading **P,** at which the weight rests, indicates the required weight of the body.

192. To graduate the common steelyard.—Let P denote the moveable weight. First let the scale pan be empty, and let **O** be the position of the weight P when the beam balances horizontally about **C.** Then the point **O** must be marked O (zero). We notice that the shorter arm **CB** and scale pan at **B** must be heavy enough for their moments about **C** to balance those of the weight of the longer arm **CA,** and of P acting at **O.**

Now let a weight W be placed in the scale pan. This weight acts on the beam at **B,** hence its moment about **C** is $W \times$ **BC.** To balance this added moment, we must increase the moment of P by an equal and opposite amount

by moving it further away from the fulcrum. Thus, if P be the new position of P, its moment about C is increased by $P \times \mathrm{CP} - P \times \mathrm{CO}$, that is, by $P \times \mathrm{OP}$. Equating

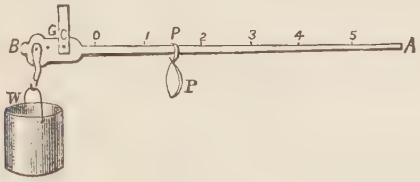


Fig. 150.

the *added* moments of P, W about C, we have therefore

$$P \times \mathrm{OP} = W \times \mathrm{BC},$$

or

$$\mathrm{OP} = \frac{W}{P} \times \mathrm{BC}.$$

Now let I denote the position of P when the unit of weight (say 1 lb.) is placed in the scale pan. Putting $W = 1$, we have

$$\mathrm{OI} = \frac{1}{P}\mathrm{BC}.$$

Therefore $\mathrm{OP} = W \,.\, \mathrm{OI}.$

Hence, if $W = 2$ units, $\mathrm{OP} = 2\mathrm{OI}$;
if $W = 3$ units, $\mathrm{OP} = 3\mathrm{OI}$;
if $W = \frac{1}{2}$ unit, $\mathrm{OP} = \frac{1}{2}\mathrm{OI}$;

and so on. We therefore have the following rule :—

Find, by actual trial, *the points* O, I *at which P must be placed when the scale pan is empty and when it contains the unit of weight, respectively. From* O *measure off on* OA *successive multiples and submultiples of the length* OI. *Their extremities will be the points of graduation for the corresponding multiples and submultiples of the unit of weight.*

193. **Modifications of the common steelyard.—Weighing machines.** In order to use the common steelyard for widely differing weights, it would be necessary either to make the arm very long, or to bring the graduations very close together : in one case the beam would be liable to bend ; in the other it would lose much of its sensitiveness. For this reason most steelyards (such as the common weighbridge or

weighing machine of railway stations) carry a scale pan attached to the longer arm at **A**, and larger weights (*e.g.* 28 lb., 56 lb., 1 cwt., etc.) are measured by a set of weights placed in this scale, smaller weights only (*e.g.* 0 to 28 lb.) being read off on the arm by the sliding weight. And, since the arm **CA** is many times longer than **CB**, each of the weights used in the scale at **A** is marked to represent a weight many times larger than itself. Thus heavy goods can be weighed by means of weights that are quite handy to lift.

This mechanical advantage is further increased in most weighing machines by means of levers placed underneath the platform carrying the goods.

In a chemical balance, small weights (milligrammes) are measured by a "rider" moved along the beam like the moveable weight in a steelyard.

194. **The Danish steelyard** (not used in this country) has no moveable weight, but the fulcrum itself is moveable, and generally consists of a loop of string from which the beam hangs. The end **A** of the beam is loaded, so that when the scale is empty it balances about a point **O** very near that end. **O** is therefore the point at which the resultant weight of the beam and scale acts.

If now a series of weights of 1 lb., 2 lb., 3 lb., etc., be placed in succession in the scale, the fulcrum will have to be moved nearer and nearer towards the end **B**, for the greater the weight at **B**, the nearer must the balancing point be to **B**. By marking on the beam **BO** the successive points about which it balances, the steelyard will be graduated in lbs., and similarly it may be graduated in grammes or for any other set of weights.

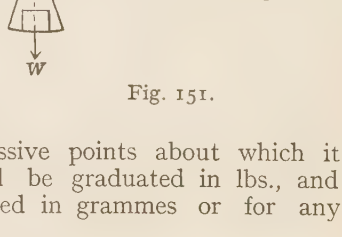


Fig. 151.

195. **To find a mathematical formula for the positions of the graduations.**

[We shall now show how to graduate the Danish steelyard *theoretically*, although, *practically*, the graduations can be found much more easily by actual trial, as explained above.]

Let W be the weight placed in the scale pan at **B**, and let P be the whole weight of the beam and scale pan acting at **O**. Then, in order

to balance, the fulcrum will have to be moved from O to a point C, such that the moments about C of P at O and W at B are equal and opposite, or $W \times BC = P \times CO$.

The graduations on the Danish steelyard are not equidistant. Supposing W to be n times P, we have

$$nBC = CO;$$

$$BO = BC + CO = (n + 1)\,BC,$$

or

$$BC = \frac{1}{n+1}\,BO.$$

Putting $n = 1, 2, 3$, we see that the distances BC corresponding to weights P, $2P$, $3P$ .. are respectively $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$ .. of BO. It readily follows that the distances between the graduations grow less and less towards the end B.

[The numbers 1, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$ .. are the reciprocals of 1, 2, 3, 4 .., and the latter numbers are in arithmetical progression. Now a series of the reciprocals of numbers in arithmetical progression is called a **harmonic progression.** Hence the distances of the successive graduations from B are in harmonic progression.]

196. **The bent lever balance** commonly used for weighing letters and small parcels consists essentially of a bent lever (usually cut out of a sheet of metal), whose arms include rather more than a right angle. The longer arm is so heavy as to rest nearly vertically in the position of equilibrium, but when a letter or parcel is suspended from the shorter arm, the moment of its weight turns the lever round. The inclination of the longer arm to the vertical in its new position of equilibrium determines the weight of the letter, and is read off by means of a pointer in front of a graduated scale arranged in various ways with which the reader is doubtless familiar. Unlike the other balances here described, the bent lever balance falls into its equilibrium position of its own accord without any adjustment of weights, etc., so that letters can be weighed very rapidly by it. A mathematical formula for the positions of the graduations can be obtained, but practically they are found by trial.

Summary of Results.

The requisites of a good balance are that it should be

(1) *true, i.e.* the beam should be horizontal when loaded with equal weights.

Conditions.—Equal arms, scale-pans of equal weight, beam properly balanced. (§ 182.)

(2) *stable, i.e.* the beam should return to its equilibrium position when displaced.

Conditions.—C.G. and middle point of beam *below* knife blade. (§ 183.)

(3) *sensitive, i.e.* the beam sensibly deflected when weights slightly unequal.

Conditions.—Height of knife blade small, arms long. (§ 184.)

(4) *rigid, i.e.* beam not bent by weights. (§ 185.)

With a *false balance* the true weight of a body may be found by *double weighing* (two methods). (§ 186.)

If the arms are unequal, W, the true weight, is the geometric mean of P, Q, the observed weights in the two pans, *i.e.* $W = \sqrt{(PQ)}$. (§ 187.)

If the scale pans are of unequal weight, the true weight is the arithmetic mean of the observed weights, *i.e.*

$$W = \tfrac{1}{2}(P + Q).$$ (§ 188.)

In the *common steelyard*, a weight is moveable along the beam ; graduations equidistant. (§§ 191, 192.)

In the *Danish steelyard*, the fulcrum is moveable, the weights fixed. (§ 194.)

EXAMPLES XIII.

(*Selected Problems* .—2, 3, 4, 5, 6, 7, 9, 11, 12, 14.)

1. Describe some form of weighing-machine, and explain carefully why the indication of the machine does not depend on the position of the body to be weighed on the platform.

2. A balance consists of a uniform rod, of length 18 in., and weight $= \frac{1}{2}$ lb., the fulcrum being $\frac{1}{8}$ in. to one side of the c.g. of the rod. If 1 lb. be in the scale attached to the shorter arm, find how much tea a customer has weighed out to him in the other scale.

3. In a steelyard the distance of the fulcrum from the point of suspension of the weight is 1 in. and the moveable weight is 6 oz. To weigh 15 lb. the moveable weight must be placed 8 in. from the fulcrum. Where must it be placed to weigh 24 lb. ?

4. The arms of a balance are 7 in. and 8 in. respectively. A body when suspended successively at the two extremities appears to weigh 4 lb. and $5\frac{1}{4}\frac{1}{9}$lb. Is the beam of the balance uniform ?

5. Can the steelyard be employed to determine whether or not the weight of a body is the same in different places ?

6. If a Danish steelyard weighs 5 lb., and if to weigh 15 lb. the fulcrum must be placed 3 in. from the point of suspension of the weight, where must it be placed in order to weigh 7 lb. ?

7. A uniform rod 2 ft. long and weighing 3 lb. is to be used as a steelyard. The fulcrum is 2 in. from one end of the rod, and the sliding weight is 1 lb. Find the greatest and the least weights which can be determined by this machine; and also where the sliding weight must be placed to indicate a weight of 20 lb.

8. Write down the weights of the least number of weights capable of weighing any exact number of pounds (i.) from 1 to 31 lb., if no weights are placed in the scale pan containing the goods, (ii.) from 1 to 40 lb., if weights are placed in either scale pan.

9. A shopman using a common steelyard alters the moveable weight for which it has been graduated. Determine whether he cheats himself or his customers.

10. A balance has a weight attached to its beam below the centre of gravity. Is the sensibility of the balance greater when the weight is hung freely by a string, or when it is rigidly attached to the balance ?

11. A Danish steelyard has the loop forming the fulcrum removed, and it is hung from a peg by two strings attached to the end of its bar. If a plumb-line be hung from the same peg, prove that the graduation on the bar which falls opposite the plumb-line will indicate the correct weight of a body placed in the scale pan.

12. A spring-balance hangs from the shorter arm of a lever, and in weighing goods, the scale pan is raised from off the ground by pulling down the longer arm of the lever and thus lifting the balance. A person stands in the scale pan and pulls himself up in this way, and the balance indicates a weight of 8 stone. If the arms of the lever are 6 in. and 2 ft. long respectively, find the man's true weight.

13. It is desired to change the moveable weight, 2 oz., of a steelyard for one of 1 lb. Show that there will be no necessity to alter the graduations, provided a weight equal to 7 times the weight of the steelyard is suspended at its centre of gravity.

14. There are no graduations on a certain Danish steelyard, and its weight is not known; but, by hanging up from the end A weights P lb. and Q lb. in succession, it is found that the corresponding distances of the fulcrum from A are a and b in. respectively. Find the position of the centre of gravity of the instrument, and show that its weight is $(bQ - aP)/(a - b)$ lb.

EXAMINATION PAPER VI.

1. Explain why in a common scale pan or letter balance it does not matter whereabouts on the pan the weights are placed; although they may be sometimes near, and sometimes further off, the fulcrum.

2. Describe the common steelyard, and show how to graduate it, and that the graduations are equidistant. What advantage is gained by the use of a steelyard?

3. Describe the Danish steelyard with fixed counterpoises, and show that the distances of the points of graduation along the load arm form a harmonical progression.

4. Explain the method of double weighing in a balance, and show that any inequality in the arms of the balance will not affect the accuracy of the result obtained.

5. A steelyard is correctly graduated when new; but, by the wearing away of the rod, the weight of the rod and the position of its centre of gravity are slightly changed. It is found that a body appearing to weigh 2 lb. in reality weighs 2 lb. $\frac{1}{2}$ oz. Find the true weight of a body appearing to weigh 10 lb.

6. Find an expression for the whole amount of work done in raising several weights through different heights.

7. A uniform beam weighs 1000 lb. and is 20 ft. long. It hangs by one end, round which it can turn freely. How many foot-pounds of work must be done to raise it from its lowest to its highest position?

8. A thread 9 ft. long has its ends fastened to the ends of a weightless rod 6 ft. long. The rod is supported in such a manner as to be capable of turning freely round a point 2 ft. from one end. A weight is placed on the thread, like a bead on a string. Give a diagram showing the position in which the rod will come to rest.

9. Find the centre of gravity of equal masses placed at each of five of the corners of a regular hexagon.

10. Two equal heavy spheres of 1 in. radius are in equilibrium within a smooth spherical cup of 3 in. radius. Show that the thrust between the cup and one of the spheres is double the thrust between the two spheres.

PART IV.

RIGID BODIES IN GENERAL.

CHAPTER XIV.

CONDITIONS OF EQUILIBRIUM OF COPLANAR FORCES.

197. DEFINITION.—By **coplanar** forces are meant *forces in one plane.* In discussing the statical effects of different systems of forces in this chapter we shall assume, unless otherwise stated, that *all the forces are applied to the same rigid body.*

Any system of coplanar forces, not in equilibrium, is equivalent either to a single force, or to a couple. (Cf. § 96.)

We will first prove that any three forces, say P, Q, R, can always be reduced to two.

Two forces can always be replaced by a single resultant, *unless they form a couple.*

Thus we can always compound P with either Q or R, unless P forms a couple with each of them.

In this case, Q and R are equal, parallel, and *like* forces (for each acts in the opposite sense to P), and thus Q and R can be compounded.

Hence any three forces can always be reduced to two.

By applying this theorem to three of the forces in any system, we should reduce the total number of forces by unity. Hence, by repeated applications of the theorem, we can obviously reduce any system of forces to two forces, and these two forces must either be in equilibrium, or form a couple, or have a single resultant.

If the system reduces to a couple, it is better to call it the *equivalent* couple, rather than the *resultant* couple, of the system.

198. Deductions from the preceding theorem.—We have proved in § 30 that the algebraic sum of the resolved parts of two forces in any direction is equal to the resolved part of their resultant in that direction.

By applying this theorem at every step in the reducing of a system of forces, we see that :—

The algebraic sum of the resolved parts of a system of coplanar forces in any direction is equal to the resolved part of their resultant force, or equivalent couple, in that direction.

It must also be noticed that the sum of the resolved parts of a couple in any direction is zero. For the resolved parts of the two forces which form the couple are equal in magnitude (since the forces are equal and are inclined at equal angles to the given direction) ; also the resolved parts are of opposite sign, since the two forces are *unlike*.

Hence, *if a system of forces is in equilibrium,* **or reduces to a couple,** *the algebraic sum of resolutes in any direction is equal to zero.*

Similarly, by repeated application of § 80 :—

The algebraic sum of the moments of any system of forces about a point is equal to the moment of their resultant force, or equivalent couple, about that point.

199. Theorem.—(i.) *If the algebraic sum of moments of a system of coplanar forces about every point in the plane is zero, the system is in equilibrium.*

For, if the system reduced to a single force, its moment would only be zero round a point in its line of action ; and, if the system reduced to a couple, the algebraic sum of moments about any point would be equal to the moment of the couple.

Similarly we have the two following theorems :—

Theorem.—(ii.) *If the algebraic sum of moments of a system is the same about every point in its plane, the system reduces to a couple.*

Theorem.—(iii.) *If the algebraic sum of moments of a system is different about different points in the plane, the system reduces to a single force.*

200. Necessary and sufficient conditions of equilibrium.

We are now in a position to prove the two following important theorems :—

Theorem.—(i.) *If the algebraic sum of moments of a coplanar system of forces vanishes about each of three points which are not in the same straight line, then the system is in equilibrium.*

It cannot reduce to a couple, for in that case the algebraic sum of moments about any point in the plane would be equal to the moment of the couple.

If possible, let it reduce to a single force ; then the moment of this force about each of the three points is zero ; therefore its line of action passes through each point, *i.e.* its line of action passes through *three points which are not in the same straight line*, which is impossible.

201. Theorem.—(ii.) *If the algebraic sum of resolved parts of a system vanishes in each of two given different directions, and if also the algebraic sum of moments vanishes about one given point, then the system is in equilibrium.*

The system cannot reduce to a couple, for in that case the algebraic sum of moments about the given point would be equal to the moment of the couple.

If possible, let it reduce to a single force. The resolved part of a force along a line is not zero unless the line of action of the force is perpendicular to the line.

Thus the line of action of the resultant force is perpendicular to each of the two given lines, and, since these are not parallel, this is impossible.

Cor.—The result of § 93 follows at once from this theorem. For the sum of the moments of these two equal and opposite couples about any point is zero.

[N.B.—To resolve a force in a given direction is to resolve it *parallel* to a given straight line. Thus the two given different directions must be determined by two given straight lines which are themselves *not parallel*, for two parallel lines are in the *same* direction and resolving parallel to the one would be the same as resolving parallel to the other.]

202. Analytical expressions for the conditions of equilibrium.—The theorem of the last paragraph may be put into a very convenient form if the two given directions are *at right angles.* Let **O** be the given point round which moments are taken, and let **OX** and **OY** be drawn in the two given directions; then $\angle$ **XOY** is a right angle.

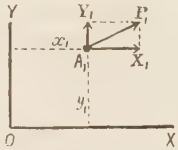


Fig. 152.

Let the forces of the system be P_1, P_2, P_3, ..., acting at the points $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$, ...; let the perpendicular distances of these points from **OY** be x_1, x_2, x_3, ..., and from **OX** be y_1, y_2, y_3, ... (cf. § 174); and let the resolved parts of the forces parallel to **OX** be X_1, X_2, X_3, ..., and parallel to **OY** be Y_1, Y_2, Y_3, ...

Then the moment of P_1 about **O**

$$= \text{algebraic sum of moments of } X_1 \text{ and } Y_1 \text{ about } \mathbf{O}$$

$$= x_1 Y_1 - y_1 X_1;$$

thus the algebraic sum of moments of the system about **O**

$$= (x_1 Y_1 - y_1 X_1) + (x_2 Y_2 - y_2 X_2) + \text{etc.}$$

This expression is most conveniently represented by the notation $\Sigma\,(xY - yX)$, where $\Sigma\,(k)$ denotes the summation of a series of terms of which k is the type.

Again, the algebraic sum of resolved parts parallel to **OX** is

$$X_1 + X_2 + X_3 + \ldots = \Sigma\,(X),$$

and of resolved parts parallel to **OY** is

$$Y_1 + Y_2 + Y_3 + \ldots = \Sigma\,(Y).$$

Thus the necessary and sufficient conditions of equilibrium in Theorem (ii.) of the last paragraph reduce to the following three equations :—

$$\Sigma\,(xY - yX) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ (1);$$

$$\Sigma\,(X) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ (2);$$

$$\Sigma\,(Y) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ (3).$$

If p, p_2, ... represent the perpendiculars from **O** on the lines of action of P_1, P_2, ..., then the moment of P_1 about **O** is P_1p_1; and thus the first condition, viz. that the sum of moments about **O** = 0, may be represented by the equation

$$\Sigma(Pp) = 0 \quad \text{.....(1A)},$$

where the signs of P_1p_1, P_2p_2, ... are determined according to the usual convention for moments.

203. Condition for a couple.—*If the algebraic sum of moments of a system of forces is the same and of the same sign when taken about each of three points not in the same straight line, then the system reduces to a couple.*

If possible, let it reduce to a single force P; and let p_1, p_2, p_3 be the perpendiculars from the three given points on the line of action of P.

Then the moment of P about each point is the same,

$$\therefore \quad Pp_1 = Pp_2 = Pp_3;$$

$$\therefore \quad p_1 = p_2 = p_3,$$

i.e. the three points are all *at the same distance* from the line of action of P. Again, since the moments are all of the same sign, the three points are all *on the same side* of the line of action of P.

Therefore the three points are in one straight line, which is contrary to hypothesis.

Examples.—(1) If forces act along the sides of a closed polygon, and are represented in magnitude and direction by the sides of the polygon taken in order, prove that they are equivalent to a couple.

Let **ABCDEF** be the polygon; join each angular point to some point **O** within the polygon.

Then moment of **AB** about **O** = $2\triangle OAB$,

" " **BC** " = $2\triangle OBC$,

and so on;

$\therefore$ algebraic sum of moments about **O** = 2 area of polygon.

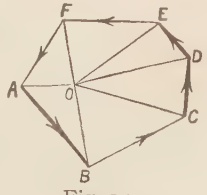


Fig. 153.

If the point O be taken *without* the polygon as in Fig. 154, we have, when the positive and negative signs of the various moments are taken into account,

algebraic sum of moments about O

$= 2\,.\,(\triangle AOB + \triangle BOC + \triangle COD - \triangle EOD - \triangle FOE - \triangle AOF)$

$= 2$ (figure OABCD — figure OAFED)

$= 2\,.$ area of polygon, as before.

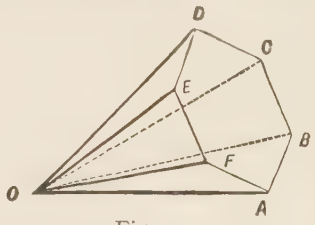


Fig. 154.

Thus the algebraic sum of moments is the same about every point in the plane; therefore, from § 203, the system is equivalent to a couple.

(2) If in a system of coplanar forces in equilibrium we alter the points of application of the forces without altering their magnitudes and directions, prove that the new system is either in equilibrium or equivalent to a couple.

For, since we have not altered the magnitudes or directions of the forces, we have not altered their resolved parts in any given direction. But, since the original system is in equilibrium, therefore the algebraic sum of resolved parts in any direction is zero in the original system, and therefore also in the new system. But if the new system were equivalent to a single force, the algebraic sum of resolved parts would not be zero, except when resolved in one direction, viz. at right angles to the line of action of the force. Hence the theorem follows.

204. To find the resultant of a given force and couple. —Let the force be Q, acting at C, and let the moment of the couple be M.

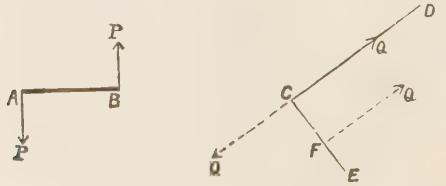


Fig. 155.

The couple can be replaced by a couple whose two forces

Q_1 and Q_2 are each equal to Q, and whose arm $= \frac{M}{Q}$ units of length. For the moment of this couple

$$= Q \times \frac{M}{Q} = M.$$

Moreover the new couple can be placed in any position in the plane. Suppose it then so placed that Q_1 acts at **C** in the opposite direction to Q. Then Q_2 will be acting in a direction parallel to Q and at a distance equal to the arm of the couple, viz. $\frac{M}{Q}$ units of length. (Fig. 155.)

In this case Q and Q_1 balance, and therefore the system reduces to Q_2. Moreover the moment of Q_2 round **C**

$$= Q_2 \times \text{arm of couple} = Q \times \frac{M}{Q} = M.$$

Thus *the resultant of the system is a force equal and parallel to the given force, and at such a distance from it that its moment about the point of application of the original force is equal to the moment of the couple.*

For the sake of clearness we have omitted to consider the sign of M. In all cases the moment of the resultant about **C** is algebraically equal to the moment of the given couple. This the student should verify.

Example.—To find the resultant of a force of 5 lb. and a couple formed of two forces of 4 lb. acting on an arm of $2\frac{1}{2}$ feet.

Let the given force of 5 lb. act at **A** in the direction represented in Fig. 156. Draw **AB** perpendicular to it.

Then, if the resultant of the three forces cuts **AB** in **B**, the algebraic sum of their moments about **B** is zero, and hence the moment of the force of 5 lb. at **A** is equal and opposite to that of the couple;

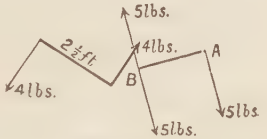


Fig. 156.

$$\therefore \quad 5 \times \mathbf{AB} = 2\tfrac{1}{2} \times 4; \quad \text{whence } \mathbf{AB} = 2 \text{ ft.}$$

At **B** introduce two equal and opposite forces of 5 lb. One of these combined with the force at **A** will form a couple whose moment is equal and opposite to the given couple, and which therefore balances the latter. Hence the resultant is the remaining force of 5 lb. acting at **B**, parallel to the original force at **A**.

205. **Two couples of equal and opposite moment, in parallel planes, balance one another.**

Firstly, let the arm **AB** of one couple be equal and parallel to the arm **CD** of the other, Fig. 157; then the forces will be equal and parallel, though not in the same plane.

Then **ABCD** is a parallelogram; therefore its diagonals **AD** and **BC** bisect each other, say at **O**.

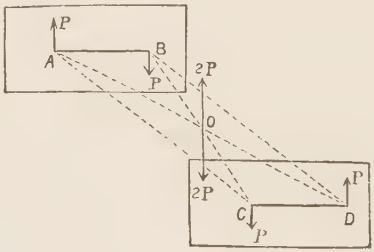


Fig. 157.

Now P at **A** and P at **D** are equivalent to $2P$ at **O**.

Also P at **B** and P at **C** are equivalent to $2P$ at **O** *in the opposite direction.*

Thus the system is in equilibrium.

Secondly, if the arms of the couple are not equal and parallel, one couple can be replaced by a new couple of equal moment in the same plane whose arm is equal and parallel to that of the other couple. Whence the theorem still holds.

Cor.—*The effect of a couple is unaltered if it be transferred to a parallel plane.*

206. **Axis of a couple.**—The statical effect of a couple on a rigid body depends only on—

(i.) The **magnitude** of its moment.

(ii.) The **direction** of the plane in which it acts. Parallel planes are, of course, to be considered as having the same direction, which fits in with the property that equal couples in parallel planes are equivalent.

(iii.) The sense in which the couple tends to turn a body.

Now all these characteristics can be represented by a straight line, for the length of the line may be made proportional to the moment of the couple, and the direction of the line may be taken perpendicular to the plane of the couple. The way in which the sense of the couple may be represented by the sense in which the line is drawn requires a little fuller consideration.

Suppose we are turning an ordinary right-handed screw (all screws in common use are right-handed) in a plank by means of a screwdriver. If we want to unscrew the screw, we apply a couple to the screwdriver tending to turn it in the opposite direction to that in which the hands of a watch move (*i.e.* the counter-clockwise direction), and this causes the screw to move *towards* us. If we applied a couple in the reverse or clockwise direction, we should drive the screw further into the board, *i.e.* should make the screw move *away* from us. Hence the sense of direction in which the screw moves depends on the sense of the couple applied to it.

The *sense* of a couple may therefore be represented by a straight line perpendicular to its plane if this line be drawn in the sense in which the couple tends to move a right-handed screw.

We now have the following definition :—

DEFINITION.—The axis of a couple is a straight line whose *direction* is perpendicular to the plane of the couple, whose *length* measures the moment of the couple, and the *sense* in which this length is measured is the sense in which the couple would tend to move a right-handed screw.

The point from which the axis of the couple is measured may be anywhere.

We may here state that, if two couples act in different planes (not parallel) on a rigid body, and if their axes be taken as two adjacent sides of a parallelogram drawn *from* their point of intersection, the couples will be equivalent to a single couple whose axis is the diagonal of the parallelogram drawn from the same point. This property, which is called the *Parallelogram of Couples*, is analogous to the Parallelogram of Forces ; as it is a little difficult to prove, we content ourselves with stating it.

Summary of Results.

Any system of coplanar forces either is in equilibrium, or or reduces to a single force or to a couple. (§ 197.)

A system of coplanar forces is in *equilibrium* if the algebraic sum of *moments* is zero about each of *three points* which are not in the same straight line. (§ 200.)

A system of coplanar forces is in *equilibrium* if the algebraic sum of *resolved parts* vanishes in each of *two different directions*, and also the algebraic sum of *moments* vanishes about *one point.* (§ 201.)

A system of coplanar forces is in equilibrium if

$$\Sigma (X) = 0,$$
$$\Sigma (Y) = 0,$$
$$\text{and } \Sigma (xY - yX) = 0. \quad (\S\ 202.)$$

A couple may always be replaced by any other couple of equal moment acting in any plane parallel to its own plane. (§ 205.)

EXAMPLES XIV.

(*Selected Problems* :—1, 3, 4, 7, 9, 10, 12, 13.)

1. Two coplanar systems of forces have equal algebraic sums of moments about each of three points which are not in the same straight line. Prove that the systems are equivalent to each other.

2. Two coplanar systems of forces have equal algebraic sums of resolved parts in each of two different directions ; also they have equal algebraic sums of moments about one point. Prove that they are equivalent to each other.

3. A system of coplanar forces, which is not in equilibrium, has its algebraic sum of moments equal to zero, about each of two points **A** and **B**. Prove that the algebraic sum of resolved parts is zero only in the direction perpendicular to **AB**.

4. If a system of forces reduces to a couple, prove that, if all the forces were applied at one point, their magnitudes and directions being unaltered, the system would be in equilibrium.

5. Prove § 95, Cor., from Question 1.

6. Find the point **O** within a triangle **ABC** such that the three forces represented by **OA, OB, OC** are in equilibrium.

7. **ABCDEF** is a regular hexagon having the following forces acting along the sides :—along **BA,** 7 lb. ; **BC,** 5 lb. ; **DC,** 1 lb. ; **ED,** 1 lb. ; **FE,** 1 lb. ; **FA,** 5 lb. Show that the system is in equilibrium.

8. Forces act along the sides of a regular hexagon **ABCDEF** as follows :—along **CB,** 5 lb. ; **CD,** 8 lb. ; **ED,** 6 lb. ; **EF,** 1 lb ; **FA,** 2 lb. Is the system in equilibrium ?

9. The following forces act on a square **ABCD** :—along **BA,** 4 lb. ; **CB,** 3 lb. ; **DC,** 2 lb. ; **AD,** 5 lb. ; **DB,** $2\sqrt{2}$ lb. Show that the system reduces to a couple, and find its moment if each side of the square measures 3 ft.

10. Given a quadrilateral **ABCD** acted on by forces *represented by* **AB, CB, DC, DA,** find the condition that it is possible to balance them by forces along the diagonals. Also find the force required along each diagonal.

11. A system of forces acts along the sides of an irregular polygon circumscribed to a circle. Prove that, if the system is in equilibrium, the sum of the forces which act in one direction round the polygon is equal to the sum of those which act in the other.

12. Prove (from § 200, or otherwise) that any coplanar system of forces which is not in equilibrium can be balanced by forces acting along the sides of a given triangle.

Prove also that, given the system and the triangle, only one set of forces will balance it.

Lastly, find the condition that the required force along any side is zero.

13. Prove, from Question 12 or otherwise, that any coplanar system of forces, not in equilibrium, can be reduced to two forces acting at two given points **A** and **B,** one of which may be inclined at any required angle to the line **AB.**

14. Prove that a system of forces can always be reduced to a force at a given point and a couple.

CHAPTER XV.

PROBLEMS ON COPLANAR FORCES.

207. In the present chapter we shall apply the theorems of the preceding chapter to the solution of problems involving the equilibrium of bodies under the action of coplanar forces.

207*a*. *Examples.*—(1) A square **ABCD** of weight W is hinged toa vertical wall at **A**. The plane of the board is vertical; it carries a weight W at **C**, and is supported with **AC** horizontal by means of a horizontal string **BE**. Determine the tension of the string and the reaction of the hinge.

The forces acting on the board are its weight W, the force W at **C**, the tension of the string T, and a reaction at the hinge which may be replaced by its unknown vertical and horizontal resolutes X and Y.

Applying § 201 :—

The algebraic sum of the vertical resolutes is zero; whence

the sum of the upward resolutes is equal to the sum of the downward resolutes.

Thus $X = W + W = 2W$..... (i.)

The algebraic sum of the horizontal resolutes is zero; whence

$T = Y$..... (ii.)

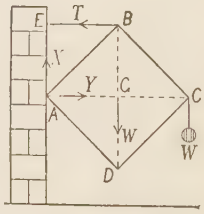


Fig. 157*a*.

The algebraic sum of the moments about **A** is zero; whence

the sum of the "counter-clockwise" moments about **A** is equal to the sum of the "clockwise" moments about **A**.

Thus $T \,.\, \mathbf{AE} = W \,.\, \mathbf{AG} + W \,.\, \mathbf{AC}$;

or, if $\mathbf{AG} = a$, $T \,.\, a = W \,.\, a + W \,.\, 2a$,

i.e. $T = 3W$..... (iii.)

X and T are given by (i.) and (iii.), and from (ii.) $Y = 3W$.

The reaction at **A** is the resultant of $2W$ along **AE** and $3W$ along **AG**.

Note the method.—The three equations derived from § 201 are sufficient to determine the three unknowns T, X, and Y. We could have resolved in any two directions and have taken moments about any one point; but the above equations are the most convenient.

(2) A uniform rod **AB** of weight 10 lb. and length 25 inches is resting on a smooth ground **DB**, leaning against a smooth vertical wall **DA**, and is prevented from slipping by a string **DC** of length 12 inches, fastened to the bottom of the wall and to a point **C** on the rod at a distance of 9 inches from its lower extremity. Find the tension of the string and the thrusts on the wall and ground.

The forces which keep the rod in equilibrium are its weight 10 lb. acting at its middle point **G**, the perpendicular reactions R, S of the wall and ground, and T the tension of the string.

We must first determine θ, the inclination of the rod to the horizontal. This inclination does not depend on the play of forces, but can be calculated from the "geometrical" data alone. For **AB** must slope at such an angle that the distance of **C** from **D** is 12 inches.

Now $\mathbf{BD} = \mathbf{AB} \cos\theta = 25 \cos\theta$.

Also, by Trigonometry,

$\mathbf{CD}^2 = \mathbf{BD}^2 + \mathbf{BC}^2 - 2\mathbf{BD} \cdot \mathbf{BC} \cos \mathbf{CBD}$.

Thus

$$144 = 625 \cos^2\theta + 81 - 450 \cos^2\theta,$$

whence $\cos\theta = \frac{3}{5}$.

It follows that $\mathbf{BD} = 25 \times \frac{3}{5}$ or 15 inches, that $\sin\theta = \frac{4}{5}$, and by Euclid I. 48 that $\angle \mathbf{BCD} = 90°$. Hence also $\angle \mathbf{CDA} = \theta$.

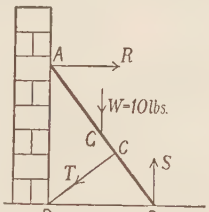


Fig. 157*b*.

Let us now apply § 201.

Equating vertical resolutes we have

$$T \cos\theta + 10 = S, \quad i.e.\ \tfrac{3}{5}T + 10 = S \quad \ldots\ldots \text{(i.)}$$

Equating horizontal resolutes,

$$T \sin\theta = R, \quad i.e.\ \tfrac{4}{5}T = R \quad \ldots\ldots\ldots \text{(ii.)}$$

Equating moments about **D**,

$$S \times \mathbf{DB} = R \times \mathbf{AD} + 10 \times \tfrac{1}{2}\mathbf{BD},$$

i.e. $$15\,S = 20\,R + 75,$$

or $$3\,S = 4R + 15 \quad \ldots\ldots\ldots\ldots\ldots \text{(iii.)}$$

These three equations are sufficient to determine the three unknowns R, S, T.

Eliminating T between (i.) and (ii.)

$$4S - 3R = 40 \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(iv.)}$$

Solving (iii.) and (iv.), we find $R = 8\frac{4}{7}$ lb., $S = 16\frac{3}{7}$ lb. Hence from (ii.) $T = 10\frac{5}{7}$ lb.

The required thrusts are equal and opposite to R and S.

(3) **AB**, **AC** are the two legs of a step ladder (each 8 ft. long), which opens out to an angle of 60° (Fig. 157*c*). The weights of the legs are respectively 20 and 40 lb., and may be supposed to act at their middle points **D** and **E**. These two points are connected by a light cord. The ladder is standing on smooth ground, and a man of weight 160 lb. is standing on a rung at **F**, two feet from **C**. Determine the tension T of the string and the reaction at the hinge **A**.

We must here consider the equilibrium of each leg separately (see Fig. 157*d*). Each leg slopes at 60° to the horizontal.

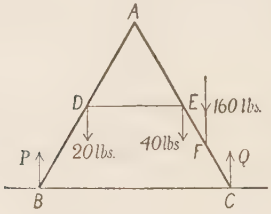


Fig. 157*c*.

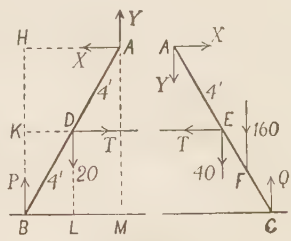


Fig. 157*d*.

The forces acting on **AB** are a vertical reaction P at **B**, the weight 20 lb. and the tension T at **D**, and the thrust at **A** from the other leg; this thrust we will replace by its (unknown) resolutes X and Y, horizontal and vertical respectively.

The forces acting on **AC** are a vertical reaction Q at **C**, downward forces of 160 lb. and 40 lb. at **F** and **E**, the tension T at **E**, and the thrust at **A** from the leg **AB**, which we will replace by its resolutes X and Y equal and opposite to those acting on the leg **AB**. [Newton's Third Law.]

We thus have five unknowns P, Q, T, X, Y. Also by applying the conditions of § 201 to each rod in turn we can obtain in all six equations; which we should expect to be more than sufficient to determine the five unknowns.

I. From the equilibrium of **AB**:—

Equating vertical resolutes, $\mathbf{P} + \mathbf{Y} = 20$ .....(i.)

Equating horizontal resolutes, $T = X$ .....(ii.)

Equating moments about **B**, $X \cdot \mathbf{BH} + Y \cdot \mathbf{BM} = T \cdot \mathbf{BK} + 20 \cdot \mathbf{BL}$,

i.e. $X \cdot 4\sqrt{3} + Y \cdot 4 = T \cdot 2\sqrt{3} + 20 \cdot 2$,

or $2X\sqrt{3} + 2Y = T \cdot \sqrt{3} + 20$ .....(iii.)

II. From the equilibrium of **AC**:—

Equating vertical resolutes, $Q = Y + 200$ .....(iv.)

Equating horizontal resolutes, $T = X$ .....(v.)

Equating moments about **C** :—

$$Y \cdot 4 + T \cdot 2\sqrt{3} + 40 \cdot 2 + 160 \cdot 1 = X \cdot 4\sqrt{3}$$

or $$2Y + T \cdot \sqrt{3} + 40 + 80 = 2X\sqrt{3} \ldots\ldots \text{(vi)}.$$

Equations (ii.) and (v.) are indentical, but we still have five equations to determine the five unknowns. We need not, however, solve completely, for equations (ii.), (iii.), (vi.) are sufficient to determine T, X, Y.

From these we find $Y = -25$, $X = \frac{70}{3}\sqrt{3}$, $T = \frac{70}{3}\sqrt{3}$.

The negative value for Y indicates that we have assumed Y in the wrong direction ; it acts downwards on **AB**, and upwards on **AC**.

The reaction at the hinge **A** is the resultant of the forces X and Y, and is easily determined by § 24.

207*b*. **Problems on Framework.**—These are very important ; the following examples will show the methods to be used.

Examples.—(1) Three equal uniform heavy rods, each of weight W, are freely jointed together to form an equilateral triangle **ABC** (see Fig. 157*e*) which is suspended from the point **A**, and carries two masses (each of weight W) suspended from the points **B** and **C** respectively. Determine the reactions at the joints.

The figure is symmetrical about the vertical through **A**, and we may therefore assume that any force on the left side of the figure is equal and symmetrical to the corresponding force on the right side of the figure.

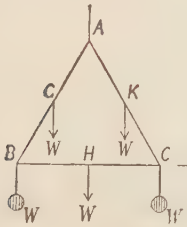


Fig. 157*e*.

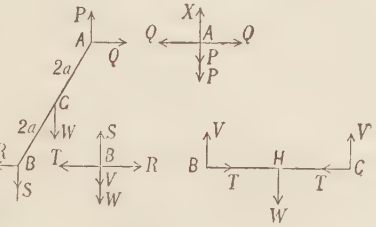


Fig. 157*f*.

At each of the points **A, B, C** we have three bodies meeting, viz. two rods and a string. We have therefore to decide in each case to which of the two rods we shall suppose the string attached. It is obvious that the different hypotheses will give us different sets of reactions.

There would be no objection to supposing that the masses at **B** and **C** are each attached to the rod **BC**. But if we suppose the string at **A** attached to either one rod or the other, we interfere unnecessarily with the symmetry of the problem. It is therefore usual in such a

case to suppose that the strings and rods at **A** do not touch one another, but are all attached to a light pivot forming the hinge ; and it is natural to make a similar assumption for the points **B** and **C**. We must then consider the equilibrium of six bodies, viz. three rods and three hinges. But the problem is considerably simplified by the symmetry.

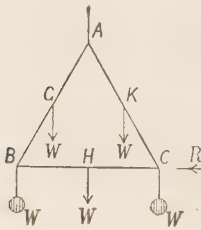


Fig. 157*g*.

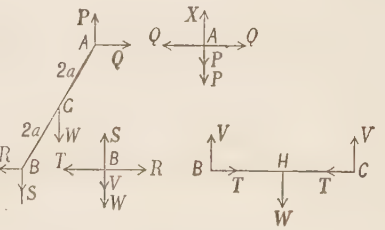


Fig. 157*h*.

Let the length of each rod be $4a$. The forces acting on the rod **AB** (Fig. 157*h*) are its weight W at G, the components P, Q of the reaction of the hinge at **A**, and the components R, S of the reaction of the hinge at **B**. The forces acting on **AC** will be equal and symmetrical with these.

The forces acting on the rod **BC** (Fig. 157*h*) will be its weight W at **H**, the components V, T of the reaction of the hinge at **B**, and the symmetrical components V, T of the reaction of the hinge at **C**.

The forces acting on the hinge at **A** (Fig. 157*h*) are X the tension of the string (obviously equal to the total weight $5W$), the components of the action of the rod **AB**, viz. P downwards and Q to the left (Newton's Third Law), and the symmetrical components of the action of the rod **AC**, viz. P *downwards* and Q to the right.

The forces acting on the hinge at **B** (Fig. 157*h*) are W the tension of the string which carries the mass ; the components of the action of the rod **BC**, viz. V downwards and T to the left, and the components of the action of the rod **AB**, viz. S upwards and R to the right.

We thus have the seven unknowns P, Q, R, S, T, V, X.

From the equilibrium of **A** we have $X = 2P$ .....(i.)

From the equilibrium of **B** we have $R = T$ .....(ii.)

and $S = V + W$ .....(iii.)

From the equilibrium of **BC** we have $W = 2V$ .....(iv.)

From the equilibrium of **AB** :—

equating vertical resolutes, $P = W + S$ .....(v.)

equating horizontal resolutes, $Q = R$.....(vi.)

equating moments about **A**, $W \cdot a + S \cdot 2a = R \cdot 2a\sqrt{3}$,

whence $W + 2S = 2R\sqrt{3}$ .....(vii.)

From (iv), $V = \frac{1}{2}W$; from (iii), $S = \frac{3}{2}W$; from (v), $P = \frac{5}{2}W$;

from (i), $X = 5W$ (as we expected); from (vii), $R = \frac{2}{3}W\sqrt{3}$; from (vi), $Q = \frac{2}{3}W\sqrt{3}$; from (ii), $T = \frac{2}{3}W\sqrt{3}$. Thus all the reactions are completely determined.

NOTE.—By the symmetry, the one half of this figure is like the reflection of the other half formed by a looking glass coinciding with the vertical through **A**. Hence if a force acts vertically up, the symmetrical force also acts vertically up; but if a force acts to the left its symmetrical force acts to the right. Do not confuse the symmetry with Newton's Third Law.

(2) Discuss the preceding problem assuming that the rods have no weight.

There are now no forces acting on the rod **AB** (Fig. 157*k*) except the reactions of the hinges at **A** and **B**. Since these two reactions are in equilibrium they must be equal and opposite and must act in the line **AB** (produced). We may therefore represent them by a force T acting at **A** in the direction **BA** produced and a force T acting at **B** in the direction **AB** produced.

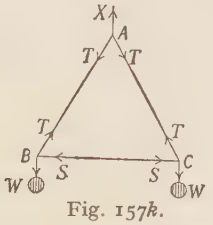


Fig. 157*k*.

Applying the Third Law, the action of the rod on the hinge at **A** is a force T acting along **AB**, and the action of the rod on the hinge at **B** is a force T acting along **BA**.

Now we shall obtain the same hypothesis (of two equa and opposite forces acting on the hinges **A** and **B**) if we regard the rod **AB** as a mere connecting link between these hinges enabling them to act on one another in accordance with the Third Law; and this method obviously effects a considerable simplification in the problem.

[Extending this reasoning for general use we derive the following useful rule:—*Whenever we have a light rod or string which is not acted upon by any forces* **excepting at its extremities,** *we may regard this rod* (*or string*) *as a connecting link capable of transmitting actions along its length in accordance with Newton's Third Law.* A rod may transmit either a "tension" or a "thrust," while a string can only transmit a "tension."]

Returning to the problem in hand :—We need now consider only the equilibrium of the three hinges. The rod **BC** must transmit a thrust S, and the rod **AC** a tension T equal to that of AB (by symmetry).

Applying Lami's Theorem to the equilibrium of the hinge A, we obtain $X : T = \sin 60° : \sin 150° = \sqrt{3} : 1$.

Also from the equilibrium of **B**, we have

$$T : S : W = \sin 90° : \sin 150° : \sin 60° = 2 : 1 : \sqrt{3}.$$

Whence we find $S = \frac{1}{3}W\sqrt{3}$, $T = \frac{2}{3}W\sqrt{3}$, $X = 2W$.

In some cases when dealing with heavy uniform rods we find it easier to replace the weight of each rod by half the weight at each extremity, and then apply the method of the preceding example. Consider the following example :—

(3) **ABCD** is a square formed by uniform rods each of weight W freely hinged together. The square is standing on its corner **C**, and is prevented from collapsing by a light string **BD**. Find the tension of this string (see Fig. 157*l*).

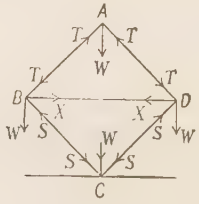


Fig. 157*l*.

Replacing each rod by half its weight at each extremity, we obtain downward forces of W at each hinge.

Consider the equilibrium of the hinge **A** (or, to be more exact, of the system at **A** which consists of the hinge together with the two particles $\frac{1}{2}W$, one at the extremity of each rod). Thrusts T along **BA** and **DA** balance the force W. Whence we find $T = \frac{1}{2}W\sqrt{2}$.

At **B** we have four forces in equilibrium, viz. the thrust T along **AB**, the tension X along **BD**, the thrust S along **CB**, and the weight W. Resolving along **AB** (to avoid introducing S) we find

$$T + W\cos 45° = X\cos 45°,$$

whence $X = W + T\sqrt{2} = 2W$.

(4) In the preceding problem determine the reaction of the hinge at **A** on the rod **AB**.

Consider the equilibrium of the particle $\frac{1}{2}W$ at the extremity **A** of the rod **AB**. It is acted upon by the thrust $T(=\frac{1}{2}W\sqrt{2})$ along **BA**, its weight $\frac{1}{2}W$, and the reaction of the hinge.

Hence the reaction of the hinge is equal and opposite to the resultant of the other two forces, and is thus easily found to be $\frac{1}{2}W$ acting horizontally to the left.

NOTE.—The method shown in Examples (3) and (4) deserves careful attention. The method of Example (3) determines correctly the tension or thrust in any *light* string or rod in the figure. The "stress" in a heavy rod is not a simple tension or thrust and varies from point to point (compare the next example). But the reactions at the extremities of the heavy rods can always be determined as in Example (4).

207*c*. **Theorem.**—*Any system of coplanar forces may be replaced by a single force acting at a given point and a couple.*

For by § 92 any force in the system may be replaced by an equal and parallel force at the given point and a couple. We thus reduce the system to a system of forces acting at the given point and a system of couples.

The system of forces acting at the given point will reduce to a single force (which may be zero), and the system of couples will reduce to a single couple (which may be zero) [§ 95].

Hence the whole system reduces to a force at the given point and a couple (either or both of which may be zero).

Example.—A uniform rod **AB** of length 6 inches and weight 3 ounces is supported in a horizontal position by a prop at each end. Determine the stress in the rod at a point **C**, 2 inches from one end.

By considering the equilibrium of the whole rod we easily deduce that the prop **A** exerts an upward force on the rod of magnitude $1\frac{1}{2}$ oz.

Now let us consider the equilibrium *of the portion* **AC**.

This portion is acted upon at **C** by a set of forces from the particles immediately to the right of **C**. These forces do not act *at one point*, for the rod must have a definite thickness, but we may regard them as acting in one plane; for if we assume that the plane of the paper divides the rod into two symmetrical halves, the forces occur in symmetrical pairs, which combine into single forces in the plane of the paper.

Hence we may replace this system of forces by a single force acting at the " point " **C**, and a couple. Let X and Y be the components of the force, and Z (not represented in the figure) the moment of the couple.

The other forces acting on **AC** are its resultant weight 1 oz. acting at its middle point **G**, and the reaction of the prop at **A**.

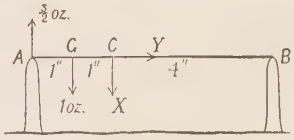


Fig. 157*m*.

We can now apply the usual conditions of equilibrium; but we must remember that the algebraical sum of resolutes for the couple Z is zero in any direction (§ 198), and that the moment of the couple is Z about any point.

Equating vertical resolutes we have $\frac{3}{2} = 1 + X$; whence $X = \frac{1}{2}$ oz. Equating horizontal resolutes, $Y = 0$. Equating moments at **C** we have

$$1 \times \mathbf{GC} + Z = \tfrac{3}{2} \times \mathbf{AC},$$

whence $Z = 2$ inch-oz. units of moment.

Thus the internal forces at **C** on the portion **AC** are equivalent to a downward force of $\frac{1}{2}$ oz. and a positive couple of 2 inch-oz. units of moment.

Note 1.—Remember the method of this problem :—(1) Consider the equilibrium of the whole rod in order to determine the external forces. (2) Consider the equilibrium of the portion of the rod in order to determine the internal forces.

Note 2.—If we collected the whole weight of the rod at its c.g. we should not alter the external forces acting on it at **A** and **B**, but we should obviously alter the nature of the *internal* stress at **C** or elsewhere.

Note 3.—The couple Z is sometimes called the " bending moment " at the point **C**. It measures the tendency of the rod to break at that point. In this problem the tendency to break is greatest at the middle point.

207*d*. **Theorem.**—*If a group of bodies is in equilibrium the external forces acting on the group will form a system of forces in equilibrium.*

We have already in the course of this book tacitly assumed that we may consider the equilibrium of a group of bodies as if the group were a single rigid body. We will now discuss the question at length, assuming only that all the forces involved may be considered coplanar.

When we have a group of bodies in equilibrium, such as a set of rods and hinges as in § 207*b*, we have (1) certain "internal forces," that is actions and reactions between the various bodies in the group, and (2) certain "external forces," that is forces due to external causes such as the weights of the rods (which are due to the attraction of the earth) or forces supplied by the points at which the group of bodies is supported, etc.

Now all the forces which act on any one body form a system in equilibrium. Hence the algebraic sum of the moments of all these forces about any point is zero.

But this is true for each body in the system. Hence by addition we see that *the algebraic sum of the moments of all the forces,* **internal and external,** *about any point is zero*.. (i).

Again, the internal forces occur in equal and opposite collinear pairs (by the Third Law). And the algebraic sum of the moments of any such pair of forces about any point is zero.

Hence by addition *the algebraic sum of the moments of all the* **internal** *forces about any point is zero* ..... (ii).

From (i) and (ii) it follows by subtraction that the algebraic sum of the moments of all the **external** forces about any point is zero, and hence (by § 199) that *the external forces would form a system in equilibrium* if applied to a single rigid body.

Example.—Use this principle to determine the reactions of the ground in Ex. 3, § 207*a*.

Using Fig. 157*c*, and treating the ladder as a rigid body, the external forces acting upon it are the weights 20 lb., 40 lb., 160lb., and the reactions P and Q.

Equating moments about **B** we obtain

$$Q \times 8 = 20 \times 2 + 40 \times 6 + 160 \times 7,$$

whence $$Q = 175 \text{ lb.}$$

Hence $$P = 20 + 40 + 160 - 175 = 45 \text{ lb.}$$

Note the general principle.—When considering the equilibrium of a rigid body we ignore the internal forces between its particles, *i.e.* the "stresses" in the body. When considering the equilibrium of a system of rigid bodies we ignore also the "internal forces" of the system, that is the mutual actions of the parts of the system.

207*e*. **Force Diagrams.—Graphic Solutions.**—We will now give examples of the method of solving problems on the equilibrium of bodies by force diagrams.

It is possible, and often very convenient, to draw the force diagrams very accurately, and obtain the required results by direct *measurement*, and not by calculation.

This method of solving problems is known as the **Graphic** method, and the theory underlying the method is known as **Graphical Statics.**

Examples.—(1) **ABCD** (Fig. 158) represents a square framework of light rods freely hinged together. **CE** is a string supporting a weight W. If the whole is suspended from **A**, determine the stress in **BD**.

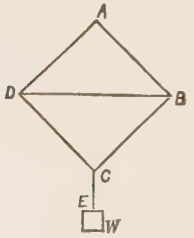


Fig. 158.

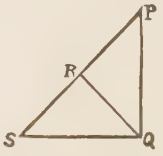


Fig. 159.

There are three forces in equilibrium at **C**, viz. W and the tensions of the rods **CD** and **CB**.

Draw a triangle **PQR** whose sides are parallel to **CE**, **CD**, and **CB**, respectively (Fig. 159). Then the sides of this triangle are proportional to the forces at **C**; thus, if **PQ** represents W, the tension along **CD** is represented by **QR**.

Again, there are three forces in equilibrium at **D**, viz. tensions along **DC** and **DA**, and a thrust along **BD**. But the tension along **DC** is equal and opposite to that at **C** along **CD**; and is therefore represented by **RQ**. Thus, if we draw **RS** and **QS** parallel to **DA** and **DB** respectively, the triangle **QRS** represents the forces at **D**; thus the thrust along **BD** is represented by **QS**.

It follows at once, by geometry, that **QS** = **PQ**.

Thus the thrust along **BD** = W.

(2) **ABCD** is a square formed of equal light rods hinged freely together; **B** and **D** are joined by a string. If equal weights are attached to each corner and the system is balancing on the corner **C**, find the tension of the string (Fig. 160).

Let W be the weight at each corner.

Draw a triangle **PQR** (Fig. 161), whose sides are parallel to **AC**, **AB**, and **AD**, respectively. Then the sides of this triangle are proportional to the forces acting at **A**; thus, if **PQ** represents W, **RP** represents the thrust along **DA**.

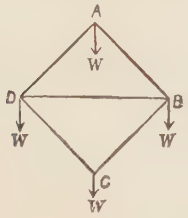


Fig. 160.

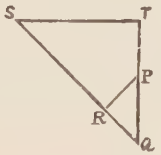


Fig. 161.

Produce **QP** to **T**, making **PT** = **PQ**.

Now there are four forces in equilibrium at **D**, of which two, viz. the weight W and the thrust down **AD**, may be represented by **TP** and **PR** respectively. Thus, if **RS** and **TS** be drawn parallel to **CD** and **DB** respectively, it follows, from the polygon of forces, that **RS** and **ST** represent the thrust along **CD** and the tension along **DB**, respectively.

The geometry of the figure gives **ST** = **TQ** = 2**PQ**; thus the force represented by **ST** is $2W$.

Hence tension of string $= 2W$.

In connection with the two preceding examples the student should note the following points:—

(*a*) If we have a system of forces in equilibrium at a point, the force polygon (or triangle) will determine the magnitude and sense of two of the forces if we know *their direction* and also know *all particulars of the remaining force or forces.* Thus in example (2) the force triangle **PQR** determines the magnitude and sense of the forces in the rods **AB** and **AD**; and then the force polygon **TPRS** determines the magnitude and sense of the forces in the rods **BD** and **CD**.

(*b*) Each triangle or polygon in the force diagram corresponds to a point in the framework, and the side common to two such figures (triangles or polygons) in the force diagram corresponds to the stress in the rod (or string) which joins the corresponding points in the framework. Thus in example (2) the triangle **PQR** corresponds to the point **A** in the framework (*i.e.* represents the forces at **A**) ; and the polygon **TPRS** corresponds to the point **D** ; while the common side **PR** represents the stress in the rod **AD.**

207*f*. **Bow's Notation.**—This is a special notation which is frequently used in framework problems ; its object is to enable us to draw the stress diagram by a mechanical rule. It is most easily explained by an example. The student should draw the figure for himself as he reads.

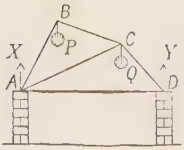


Fig. 161*a*.

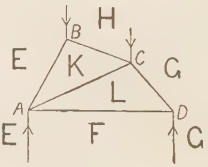


Fig. 161*b*.

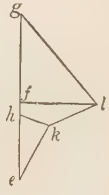


Fig. 161*c*.

Example.—**ABCD** is a framework of light rods (Fig. 161*a*) resting on two supports at **A** and **D**, and carrying two weights P and Q at the hinges **B** and **C**. It is required to calculate the stresses in the rods and the thrusts X and Y at **A** and **D**.

We will first redraw the framework (Fig. 161*b*) representing only the lines of action of the external forces and the framework itself (which gives the lines of action of the internal forces). The lines of action of P, Q, X, Y are drawn outside the framework to make a clearer figure.

In Fig. 161*b* we letter the *areas* **E, F, G, H, K, L** which are marked off from one another by the lines of the figure* ; and we name any force in the figure by quoting the two areas which are separated by its line of action. Thus the stress in **AC** is now quoted as the force "**KL**" or "**LK**," and the weight Q as the force "**HG**" or "**GH**."

* In the diagram some of these areas have been marked in more than one place.

We naturally start with the equilibrium at **B**, since at this point the only unknowns are the magnitudes of two of the forces.

The forces at **B** are **HE, EK, KH** (quoting the letters in counter-clockwise rotation). Draw *he* to represent **P**, *ek* parallel to **AB**, and *hk* parallel to **BC** (Fig. 161*c*). Then *he, ek, kh* represent the forces **HE, EK, KH.**

The forces at **C** are **GH, HK, KL, LG** (quoting the letters in counter-clockwise rotation) of which the first two are now known. Hence if we draw *gh* to represent **GH** or **Q**, we can complete the force polygon *ghkl*, and the sides *kl, lg* represent the forces **KL, LG.**

The forces at **D** are **GL, LF, FG**, and since **GL** is represented by *gl* we can complete the force triangle *glf*.

The figure is now complete, for the forces at **A** are **KE, EF, FL, LK,** which are represented by the links *ke, ef, fl, lk*.

NOTE 1.—Each *area* in the framework corresponds to a *point* in the stress diagram.

NOTE 2.—The two forces **KL** and **LK** are of opposite sense, but the notation takes no count as to which of the two acts to the right and which to the left. When we are dealing with the equilibrium at any point we quote each force by the adjacent areas taken in *counter-clockwise rotation* round the point.

NOTE 3.—Bow's Notation does not apply if there are forces on the rods at other points than the hinges.

207*g*. **Graphic method for determining the resultant of coplanar forces.**—In Fig. 161*d* let *P, Q, R, S* be the coplanar forces whose resultant is required. (The student should draw the figure for himself as he reads.)

Construction.—(*a*) Make a force diagram **ABCDE** (Fig. 161*e*), in which **AB, BC, CD, DE** represent *P, Q, R, S* in magnitude and direction in the usual manner. *Then* **AE** *shall represent the resultant X in magnitude and direction.*

(*b*) Take any point **O** in the force diagram, and join it to the angular points **A, B, C, D, E.** In Fig. 161*d*, starting from *any* convenient point **F**, draw **FG, GH, HK, KL, LM** parallel to **AO, BO, CO, DO, EO** respectively, so that **G, H, K, L** lie on the lines of action of *P, Q, R, S* respectively; [*i.e.* draw **FG** parallel to **AO** meeting the line of action of *P* in **G**, draw **GH** parallel to **BO** meeting the line of action of *Q* in **H**, etc.] Then **N**, the point of intersection of **FG** and **LM**, *shall be a point on the line of action of the resultant,* and thus the resultant is completely determined.

Proof.—The force P acting at **G** can be resolved into two forces acting along **FG** and **HG**; and this can be done by means of the triangle **AOB**, whose sides are respectively parallel to these three lines of action. Thus the required components of P are represented in magnitude and direction by **AO**, **OB**. Hence we have

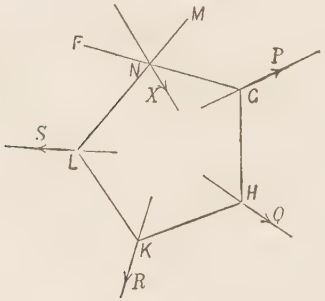


Fig. 161*d*.

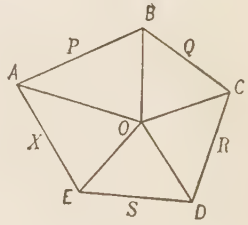


Fig. 161*e*.

$$P = \begin{cases} \mathbf{AO} \text{ acting along } \mathbf{NG} \ldots\ldots\ldots & \text{(i.)} \\ \mathbf{OB} \quad ,, \quad ,, \quad \mathbf{HG} \ldots\ldots\ldots & \text{(ii.)} \end{cases}$$

Similarly

$$Q = \begin{cases} \mathbf{BO} \quad ,, \quad ,, \quad \mathbf{GH} \ldots\ldots\ldots & \text{(iii.)} \\ \mathbf{OC} \quad ,, \quad ,, \quad \mathbf{KH} \ldots\ldots\ldots & \text{(iv.)} \end{cases}$$

$$R = \begin{cases} \mathbf{CO} \quad ,, \quad ,, \quad \mathbf{HK} \ldots\ldots\ldots & \text{(v.)} \\ \mathbf{OD} \quad ,, \quad ,, \quad \mathbf{LK} \ldots\ldots\ldots & \text{(vi.)} \end{cases}$$

$$S = \begin{cases} \mathbf{DO} \quad ,, \quad ,, \quad \mathbf{KL} \ldots\ldots\ldots & \text{(vii.)} \\ \mathbf{OE} \quad ,, \quad ,, \quad \mathbf{NL} \ldots\ldots\ldots & \text{(viii).} \end{cases}$$

Now of these eight forces, six cancel in pairs, viz. (ii) and (iii), (iv) and (v), (vi) and (vii). Hence the original system is equivalent to the other two forces, viz. **AO** and **OE** acting along **NG** and **NL** respectively.

Using the triangle **AOE** we see that their resultant is equivalent to the single force X acting at **N**.

NOTE 1.—The conditions that P, Q, R, S form a system in equilibrium are that (*a*) **A** and **E** should coincide, (*b*) **FG** and **LM** should coincide.

NOTE 2.—The method can be applied to a system of parallel forces, in which case **ABCDE** will be a straight line.

NOTE 3.—The polygon **NGHKL** is called the "link polygon," as it links together the lines of action of the forces. Note the correspondence between the sides of the link polygon and the lines in the stress diagram to which they are parallel,—**GH** is parallel to **BO**, but **GH** links the *lines of action* of P and Q, while **B** is the intersection of the lines *representing* P and Q.

SUMMARY.

For a *single body* under the action of coplanar forces apply the conditions of § 201. (§ 207*a*).

For a *system of bodies* apply the conditions of § 201 and Newton's Third Law. (§ 207*a*).

In a *framework* problem if no forces act on the rods excepting at their extremities we may deal with the equilibrium of the hinges only. (§ 207*b*).

In a *framework of heavy rods* we may sometimes replace the weight of each rod by half the weight at each extremity. (§ 207*b*).

Any system of coplanar forces may be replaced by a single force acting at a given point and a couple. (§ 207*c*).

If a group of bodies is in equilibrium the external forces acting on the group will form a system of forces in equilibrium. (§ 207*d*).

Graphic Methods, and Bow's Notation. (§ 207*f*, *g*).

EXAMPLES XV.

(*Selected Problems* :—1, 4, 8, 10, 12, 15, 18, 20, 24, 25, 26, 28, 29, 30, 32, 33, 35, 36, 38, 40, 41, 42 , 44, 45.)

1. A ladder rests against a smooth wall, the ground being also smooth. Compare the horizontal forces which must be applied to the bottom of the ladder to preserve equilibrium when a weight equal to the weight of the ladder is placed on the ladder at the bottom and top respectively.

2. A ladder, with its centre of gravity three-eighths of the way up it, stands on a smooth horizontal plane resting against a smooth vertical wall, and the middle point is tied to a point in the wall by a horizontal rope. Find the tension of the rope in terms of the weight, W, and length l, of the ladder, and the length, a, of the rope.

3. A uniform rod of weight W rests with its lower end **A** and its upper end **B** in contact respectively with smooth horizontal and vertical walls, and one point **C** is attached by a string to **O** the point of junction of the walls. Show that if the rod rests at an inclination α to the horizon the tension of the string is

$$\tfrac{1}{2} W \cos \alpha / \sin (\alpha - \theta) ;$$

where $$\tan \theta = \mathbf{AC} \tan \alpha / \mathbf{BC}.$$

4. Prove that if **E, F** are the feet of the perpendiculars from **B, C** upon the opposite sides of a triangle **ABC**, a force P acting along **EF** can be replaced by forces $P \cos A$, $P \cos B$, $P \cos C$ acting along the sides.

5. Forces act along the sides **AB, BC, CD, DA** of a plane quadrilateral taken in order, and their magnitudes are p, q, r, s times the length of the sides in which they act. Prove that they are equivalent to a couple if

$$(p - q)\ \mathbf{OB} = (r - s)\ \mathbf{OD},$$

and $$(q - r)\ \mathbf{OC} = (s - p)\ \mathbf{OA},$$

where **O** is the intersection of **AC** and **BD**.

6. A smooth lamina in the form of an equilateral triangle **ABC** is placed on a horizontal table so that each of its sides touches one of three smooth pegs **P, Q, R**, where $\mathbf{AR} : \mathbf{RB} = \mathbf{BP} : \mathbf{PC} = \mathbf{CQ} : \mathbf{QA} = n : 1$. A couple G acts on the lamina. Show that the pressure on each of the pegs is

$$\frac{2}{3} \cdot \frac{n + 1}{n - 1} \cdot \frac{G}{\mathbf{BC}}.$$

7. Points **D, E, F** are taken in the sides **BC, CA, AB** of a triangle so that $\mathbf{BD} : \mathbf{DC} = \mathbf{CE} : \mathbf{EA} = \mathbf{AF} : \mathbf{FB} = m : n$. Show that forces completely represented by **AD, BE, CF** are equivalent to a couple whose moment is represented by twice the area

$$\mathbf{ADC} + \mathbf{BEA} + \mathbf{CFB} - (m + 2n)\ \mathbf{ABC}/(m + n).$$

8. Two equal heavy rods, of weight W and length $2a$, are freely hinged together, and placed symmetrically over a smooth fixed sphere of radius r. Show that the inclination of each rod to the horizontal is given by $r \tan^3\theta + r \tan\theta = a$.

9. **AB** and **BC** are two equal rigid rods hinged at **B**; at the ends **A** and **C** are smooth rings which are free to slide on a fixed horizontal rod **AC**, and the joint **B** is supported by a string **BD**, to whose other end is attached a smooth ring free to slide on **AC**. Find its tension, if the weight of each rod is W.

*10. Two equal uniform bars, each of weight W, are placed in a vertical plane with their upper ends joined by a smooth heavy hinge, whose weight is half that of either bar, and their lower ends resting on a smooth horizontal plane and connected by a string. The bars and string form an equilateral triangle. Find the tension of the string, and the direction and magnitude of the action of the hinge on each bar.

*11. Two equal rods, **AB, AC,** freely joined at **A,** have their ends **B, C,** connected by a light string. **B** and **C** rest symmetrically on two smooth planes inclined at angles α to the horizon, with their slopes facing one another, the plane of the rods being vertical. Show that the tension of the string is $\frac{1}{2} W (\tan\theta - 2\tan\alpha)$, W being the weight of a rod, and 2θ the angle **BAC.**

*12. Two equal weightless rods **AB, AC,** of length a, are hinged at **A.** A weight W is attached at **A,** and equal weights w at **B** and **C.** The system is placed symmetrically with each rod resting against one of two pegs in a horizontal line, at a distance $2c$ apart, and with **A** vertically above the middle point of the line joining the pegs. Prove that equilibrium is possible, if

W/w is not greater than $2(a - c)/c$;

and find the angle which the rods make with the vertical.

*13. A rod of length $2a$ is resting in contact with a smooth sphere of radius r, and the lower end of the rod presses against a smooth wall; if the distance of the centre of the sphere from the wall is c, and the rod is inclined at an angle α to the wall, prove that

$$a \sin^3\alpha + r \cos\alpha = c.$$

---

* Questions marked with an asterisk are good examples on Force Diagrams, but they can, of course, be solved by other methods.

*14. Four rods smoothly jointed together form a parallelogram **ABCD**; strings connect **A** with **C**, and **B** with **D**. Show that their tensions are in the ratio of their lengths.

*15. A rhombus formed of light rods jointed together is hung up by one corner. A weight W is suspended from each of the three lower corners, and the shape of the rhombus is preserved by a light rod along the horizontal diagonal. Find the tension or thrust in each of the rods.

*16. Four equal rods hinged together form a rhombus **ABCD**, which is hung in a vertical plane from a fixed point **E**, by two equal rods **EB** and **ED** and a third **EA**, of such lengths that the angle at **C** is 60°, and each of the angles at **E** is 15°. The rods being without weight, and a weight W attached to **C**, prove that the force in the rod **EA** is

$$W(\sqrt{3} + 1).$$

*17. Three equal weightless rods **AB, BC, CD** are jointed together smoothly at **B** and **C**, whilst **A** and **D** are fixed at a distance apart equal to the length of either rod. **B** and **C** are fastened by strings to points **E** and **F** in the same straight line with **AD** such that **EA** = **AD** = **DF**. Show that the tensions in these strings are proportional to their lengths.

*18. Two equal uniform heavy rods **AH, BH**, are hinged to points **A** and **B** in the same vertical line, and hinged together at **H**. If each rod is inclined at an angle α to the horizontal, and each rod weighs W lb., find the magnitude and direction of the reaction at **H**.

*19. **ABCD** is a rhombus formed by four equal uniform rods freely jointed together, and is in equilibrium with **A** resting on a horizontal plane, and **AC** vertical, and a string connects **B, D**. Show that the tension of the string is $2W \tan \theta$, where W is the weight of a rod, and θ its inclination to the vertical.

*20. **AB, BC, CD** are three rods of equal length, and of weights w, W, w respectively. They are smoothly hinged at **B** and **C**; and at **A** and **D** are smooth rings which support the rods on a horizontal bar. Find the horizontal force which must act on each ring in order to keep the three rods in the form of half a regular hexagon.

*21. A regular hexagon **ABCDEF** is formed by six equal uniform jointed rods, and is kept in shape by two weightless wires joining **B**

to **F**, and **C** to **E**. Find the force along these wires when the hexagon is suspended from **A**.

*22. Two equal light rods **AB**, **BC** are jointed at **B**, and have their middle points connected by an inelastic string of such a length that when it is straightened the angle **ABC** is a right angle. If the system is suspended from **A**, and a weight W attached at **C**, find the tension of the string, and the reaction at the hinge.

*23. Two equal light rods **AB**, **BC**, each 5 in. long, are connected by a string 5 in. long, which is attached to **AB** at a point 2 in. from **A**, and to **BC** at a point 1 in. from **C**. If the system is suspended from **A**, and carries a weight W at **C**, find the tension of the string.

*24. Two equal uniform rods **AB**, **BC**, each of weight W, are jointed at **B**, and have their middle points connected by an inelastic string of such a length that when it is straightened, the angle **ABC** is a right angle. If the system is suspended by a string attached to the point **A**, prove that the inclination of **AB** to the vertical will be $\cot^{-1}3$, and find the tension of the former string and the action at the joint.

*25. Four equal uniform rods are freely jointed together so as to form a square **ABCD**, and the system hangs from the point **A**, the square form being maintained by an inextensible string connecting the middle points of **AB** and **BC**. Prove that if W is the weight of each rod, the tension of the string is $4W$.

*26. Three equal uniform rods, each of weight W, are hinged together to form an equilateral triangle. If the system be suspended by the middle point of one of the rods, the stress at the lowest angle will be $W/2\sqrt{3}$, and that at each of the upper angles $W\sqrt{13/2}\sqrt{3}$.

*27. An equilateral triangle made of uniform rods, each of weight W, freely hinged at their extremities, is suspended from one angular point. Find the magnitude and direction of the reaction at either of the other angular points.

28. A gate 5 ft. high and 8 ft. broad weighs 100 lb. : it is supported by two hinges a foot from the bottom and top respectively. The lower hinge can only exert a horizontal reaction. Find the reactions at both hinges, if a boy of weight 60 lb. is sitting on the end of the gate.

29. A square board whose side measures a inches is in equilibrium in a vertical plane resting on two smooth pegs which are at the same horizontal level, a inches apart; the board carries a weight at one corner, and two of its sides are inclined at 30° to the horizontal. Find the ratio of the weight at the corner to the weight of the board.

30. A triangular board **ABC** is lying on a smooth horizontal table. It is acted on by a couple of moment 2 ft.-lb. units, but is prevented from moving by three smooth pegs **D, E, F,** which are in contact with the sides **BC, CA, AB** respectively. Find the pressures on these pegs, if **AB** = 3 in., **BC** = 4 in., **CA** = 5 in., **AF** = **BD** = **CE** = 1 in.

31. A square board **ABCD** is placed on a smooth plane inclined at 60° to the horizontal, so that the plane of the board is vertical and **CD** is in contact with the inclined plane. Its highest corner **A** is attached to the plane by a horizontal string. Prove that it cannot rest in equilibrium in this position. If the weight of the board is W, find the least weight which must be suspended from the point **B** to render equilibrium possible.

32. Determine the condition of equilibrium of a body moveable about an axle.

Prove that the pull exerted by a man in a tug of war is $\frac{a}{b}$ times his weight, when a is the horizontal overhang of his C.G. backwards over his heels, and b is the height of the rope above the ground.

33. Show that a weight of W tons suspended from the point **C** of a crane **ABC** against a wall **AB,** in which the bar **BC** is horizontal and **A** vertically above **B,** will set up a thrust of $\frac{BC}{AB}W$ tons in **BC,** and a pull of $\frac{AC}{AB}W$ tons in **AC.**

If the weight W is being hoisted by a chain passing up through a pulley at **C,** and coming down again at an angle β to the horizon, prove that the additional thrust in **BC** and pull in **AC** are respectively

$$\frac{\sin(\alpha+\beta)}{\sin\alpha}W \text{ and } \frac{\sin\beta}{\sin\alpha}W,$$

α denoting the angle **ACB.**

34. A circular ring of radius a and weight W is suspended in a horizontal position by three vertical threads each of length l, attached to points on its circumference; prove that in order to hold the ring horizontal and twisted through an angle θ, a horizontal couple must be applied to it equal to

$$W \frac{a^2 \sin\theta}{\left(l^2 - 4a^2 \sin^2 \frac{\theta}{2}\right)^{\frac{1}{2}}}.$$

35. Draw a triangle **ABC**, having the vertical angle **A** large and the base **BC** horizontal; produce **BC** to **D**. Let **AB** denote a rod connected by smooth hinges to a fixed point **B** and to the end **A** of a rod **AC** whose other end **C** can move in a smooth groove **BCD**: the weight of the rods being negligible a force F is applied in the plane **ABC** at right angles to **AB** at **A**: find an expression for the force transmitted by the rod **AC** along the groove.

Also find the force transmitted if **F** be applied at **A** perpendicular to **BC**.

*36. A triangle **ABC**, formed of three rods jointed together, is lying on a horizontal plane. Taking a point **P** in **AB**, and a point **Q** in **AC**, and supposing the points **P** and **Q** connected by a tightened string, find by a geometrical construction the direction of the stress at the joint **A**.

*37. Draw the stress diagram of a span of a symmetrical bridge, in which the roadway is horizontal and divided into equal bays, supported by light steel ropes, attached to the top of the nearest pier, the load being supposed concentrated uniformly at the points of attachment to the roadway.

Draw the diagram for any intermediate stage of the construction, the bridge being built out from each pier.

[Assume that there is an odd number of bays and the tension in the middle bay is zero.]

*38. Draw an irregular polygon **ABCDEF**; let **ABCDEF** represent a string fastened at **A** and **F** and acted on by forces P, Q, R, S acting outwards from the polygon at **B, C, D, E** respectively. Show by means of a force diagram that if we assume the shape of the polygon and the lines of action of the forces we can determine the ratios of the forces. [Bow's notation can be used.]

*39. **A** and **F** are two points joined by a string whose length is greater than **AF**. At intermediate points **B, C, D, E** of the string weights are attached, each equal to W. Assuming that each section of the string slopes upwards to the right, show by means of a force diagram that the tangents of the inclinations of the strings to the horizon are in Arithmetical Progression. [Bow's notation can be used].

*40. Draw accurately an isosceles right-angled triangle **ABC**, right-angled at **B**. Take a point **D** within the triangle such that the angles **DAC, DCA** are each 30°. **ABC** is a triangle formed of freely jointed rods; **DA, DB, DC** are stretched strings. The tension in **DA** is 1 lb. Draw a stress diagram, and, by measurement from it, determine the stresses in the other two strings and the three rods.

*41. Draw accurately an equilateral triangle **ABC**, with **BC** horizontal and **A** above **BC**; on the side **AC** remote from **B** describe a right-angled isosceles triangle **ACD**, right-angled at **C**. The figure represents a light framework resting on props at **B** and **C**, and carrying weights 8 and 2 lb. at **A** and **D** respectively. Determine by measurement from a stress diagram the stress in **AC**, and the pressure on the prop at **B**.

*42. Draw a vertical line **AB**, **A** above **B**. On the same side of **AB** draw lines **BC, BD** making angles of 40° and 73° respectively with the upward vertical, and meeting the horizontal through **A** in **C** and **D**. If the figure represents a light framework fastened to a vertical post at **A** and **B** and supporting loads of 1 and 2 tons at the hinges **C** and **D** respectively, determine by measurement from a force diagram the stresses in **AC** and **BC**.

*43. Solve Question 12 in Examples VII. by the graphic method of § 207*e*.

*44. Solve Question 4 in Examples X. by the graphic method of § 207*e*.

*45. Draw a square **ABCD** of side half an inch; suppose that forces of 4, 5, 6, 7 oz. wt. act along the sides **AB, BC, CD, DA** respectively. Determine by a force diagram at what distances from **A** the line of action of the resultant cuts **BA** and **AD** produced.

CHAPTER XVI.

THE PRINCIPLES OF VIRTUAL WORK AND VIRTUAL VELOCITIES.

208. Virtual Work.—We often find it convenient for the sake of argument to *suppose* a particle displaced from one position **A** to another position **B**, although the particle may have no tendency to move in this direction, or, perhaps, in any direction.

In this case **AB** is called the **virtual displacement** of the particle ; and the work which *would* be done by any force acting on the particle during the displacement is called the **virtual work** of that force.

The results of §§ 38, 39 can now be stated thus :—

If a particle under the action of any forces receive a virtual displacement, then the algebraic sum of the virtual works of the forces is equal to the virtual work of their resultant.

And, if the system of forces be in equilibrium, the algebraic sum of the virtual works is zero.

The last property is a case of what is called the **Principle of Virtual Work.**

209. Converse of the Principle of Virtual Work for a particle.—*If a particle is acted on by a system of forces in one plane, and if the virtual work of the system vanishes for each of two displacements which are not in the same straight line, then the system is in equilibrium.*

For, if not, the virtual work of the resultant must also vanish for each displacement. But

the virtual work of the resultant = displacement × resolved part of the resultant along the displacement ;

∴ resolved part of resultant along each displacement = 0 ;

therefore the line of action of the resultant is perpendicular

to each displacement, which is impossible since the displacements are not in the same straight line and the resultant is in the same plane with them both.

210. **Application to the Inclined Plane.**—As an example of the application of the Principle of Virtual Work, let us consider the case of a particle supported on a smooth inclined plane by a force acting along the plane.

(1) To find the relation between P and W.

Give the particle a virtual displacement, **OD**, along the plane. Draw **Dd** perpendicular to the line of action of W produced. Then, since **OD** is perpendicular to R,

$\therefore$ work done by $R = 0$;

also work done by $P = P \,.\, \mathbf{OD}$;

and work done by $W = -\, W \,.\, \mathbf{Od}$;

whence the Equation of Virtual Work gives

$$P \,.\, \mathbf{OD} - W \,.\, \mathbf{Od} = 0;$$

$$\therefore \quad \frac{P}{W} = \frac{\mathbf{Od}}{\mathbf{OD}} = \frac{\mathbf{CB}}{\mathbf{AC}} \text{ (by similar triangles).}$$

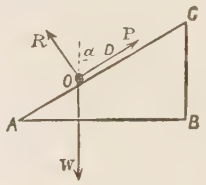


Fig. 162.

(2) To find the relation between P and R.

Give the particle a virtual displacement, **OD**, horizontally. (This displacement could *not* be actually made in one direction without breaking the plane, or in the opposite direction without the particle leaving the plane.)

Draw **Dd** and **De** perpendicular to the lines of action of P and R, respectively. Then, since W is perpendicular to **OD**,

work done by $W = 0$;

also work done by $P = +\, P \,.\, \mathbf{Od}$;

and work done by $R = -\, R \,.\, \mathbf{Oe}$;

whence the Equation of Virtual Work gives

$$P \,.\, \mathbf{Od} - R \,.\, \mathbf{Oe} = 0,$$

$$\therefore \quad \frac{P}{R} = \frac{\mathbf{Oe}}{\mathbf{Od}} = \frac{\mathbf{Oe}}{\mathbf{De}} = \frac{\mathbf{CB}}{\mathbf{AB}}$$

(by similar triangles).

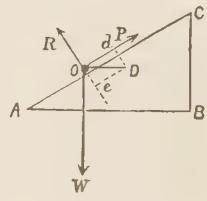


Fig. 163.

N.B.—It should be noticed that, to find the relation between P and W, we take a virtual displacement perpendicular to R; then the virtual work of $R = 0$, and therefore R does not enter into the Equation of Virtual Work. Similarly, in (2) we take a virtual displacement perpendicular to W.

211. **Virtual Velocities.**—Suppose a particle under the action of a system of forces in equilibrium to be moved in any manner whatever; then, by the principle of virtual work, the work done by the system during any time is zero; therefore the *rate of doing work* of the system at any instant is zero.

Now, the rate of doing work at any instant depends only on the magnitude and direction of the forces at that instant. Hence: —

If a particle under the action of any system of forces is in motion and passes through an equilibrium position, the algebraic sum of the rates of working of the forces at that instant is zero.

As an illustration, we may quote the case of a simple pendulum. The equilibrium position is the lowest point of the swing; here the velocity is horizontal, and hence neither the tension of the string or gravity is doing any work as these are both vertical forces.

The above theorem is the "**Principle of Virtual Velocities**" for a single particle. In this form it is of little practical value, but we shall show in this chapter that the principle can be extended to systems of particles or of rigid bodies, and is then most useful.

Meanwhile it is necessary to find a formula for the rate of working of a force. Let P be a force whose point of application is travelling with a velocity u, and let θ be the angle between the directions of the force and the velocity.

The *rate of working* is measured by the work that would be done in unit time if the velocity were unchanged, *i.e.* if the point of application moved a distance u in a direction inclined at an angle θ to the direction of the force. Since the projection of this displacement u along the line of action of P is $u \cos \theta$, the work done $= Pu \cos \theta$.

Hence **rate of working** $= \mathbf{Pu \cos \theta}$ .....(1).

We can now express the Principle of Virtual Velocities as a formula. For if P_1, P_2, P_3, etc., be the force acting on the particle and θ_1, θ_2, θ_3, etc., the inclinations of these forces to the direction of the velocity u when passing through the equilibrium position, we have

$$P_1 u \cos \theta_1 + P_2 u \cos \theta_2 + P_3 u \cos \theta_3 + \text{etc.} = 0,$$

i.e. $$\Sigma(\boldsymbol{Pu} \cos \theta) = 0 \quad \ldots\ldots\ldots\ldots\ldots (2).$$

This is the **Equation of Virtual Velocities.**

Note the difference between the Principle of Virtual Velocities and the Principle of Virtual Work. The latter principle asserts that the sum of the works done by a system of forces is zero for any displacement of any magnitude whatever, *provided that the forces continue to remain in equilibrium during the whole of the displacement.* In most statical problems the forces do *not* remain in equilibrium in the displaced position, and then the principle only holds for a *very small displacement.* But for such a displacement the virtual works of the forces are themselves very small, and it is therefore more convenient to use the principle as stated above.

212. **On the motion of a right line of constant length in a plane.**—Let one end **A** of a line **AB** be fixed ; then, if the length of the line is fixed, the only motion possible for **B** is to move round the circle of which **A** is the centre and **AB** the radius ; and, therefore, the velocity of **B** at any instant must be perpendicular to the position of **AB** at that instant. Now, if **A** is also free to move in the plane, it will still be true that, in whatever way the line moves, **B**'s path *relative* to **A** is a circle or part of a circle ; therefore **B**'s velocity *relative* to **A** at any instant is at right angles to the position of **AB** at that instant. Again, **B**'s actual velocity at any instant is compounded of its velocity relative to **A** and **A**'s actual velocity.

But the resolved part of a resultant velocity in any direction is equal to the algebraic sum of the resolved parts of the component velocities. (Cf. § 30 for the corresponding theorem in forces.)

$\therefore$ resolved part along **AB** of **B**'s actual velocity
= resolved part of **B**'s velocity relative to **A**
+ resolved part of **A**'s actual velocity.

But we have proved that **B**'s velocity relative to **A** is perpendicular to **AB** ; and, therefore, its resolved part along **AB** is zero ;

$\therefore$ resolved part of **B**'s velocity along **AB**
= resolved part of **A**'s velocity along **AB**.

Hence, *if a line* **AB** *of constant length be moving in a plane in any manner, the resolved parts of the velocities of* **A** *and* **B** *along* **AB** *are always equal to each other.*

213. **Principle of Virtual Velocities for a network of particles.**—Now consider a system of particles (*e.g.* three particles **A, B, C**) in a plane, rigidly connected with each other by light wires, and acted on by a system of forces which keep them in equilibrium. Then each wire will probably be in a state of strain, *i.e.* it may be pulling the two particles to which it is attached towards each other (in which case we say there is a "*tension*" along the wire); or it may be pushing them away from each other (in which case we say there is a "*thrust*" along the wire).

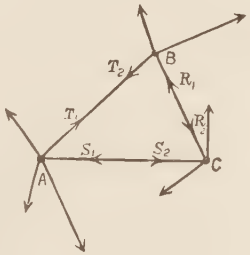


Fig. 164.

[These tensions and thrusts, which are merely reactions along different parts of the system, are called the **internal** forces of the system; and the other forces, which are due to actions from outside, are called **external** forces.]

Since the system is in equilibrium, each particle is in equilibrium; therefore, if we give virtual velocities to **A, B,** and **C,** algebraic sum of rates of working of the forces acting on **A** = 0; and similarly for **B** and **C.**

Therefore, by summing,

algebraic sum of rates of working of all the forces (external and internal) = 0 .....(1).

This result is true *no matter what virtual velocities* we give to **A, B,** and **C.**

But now suppose that we limit ourselves to giving only such virtual velocities to **A, B,** and **C** *as they can respectively have without breaking the wires* (say, u, v, and w). Also let R_1, R_2, S_1, S_2, T_1, T_2 be the actions and reactions along the wires. Then, by § 212,

resolved part of u along **AB** = resolved part of v along **AB.**

Moreover, by Newton's Third Law,

$$T_1 = -T_2;$$

but rate of working of $T_1 = T_1 \times$ resolved part of u along **AB,**

rate of working of $T_2 = T_2 \times$ resolved part of v along **AB**,
$= -T_1 \times$ resolved part of u along **AB**;

$\therefore$ algebraic sum of rates of working of T_1 and $T_2 = 0$.

Similarly for R_1 and R_2, S_1 and S_2;

$\therefore$ *sum of rates of working of all the internal forces* $= 0$ ..... (2);

therefore, by subtraction from (1),

algebraic sum of rates of working of all the external forces $= 0$ ..... (3).

214. The above theorem can obviously be extended to a system of any number of coplanar particles rigidly connected; also to two particles connected by an inextensible string; and we may further extend it to the case of a rigid lamina, or, indeed, to a rigid body of any shape. For, whatever be the nature of the forces which keep the various particles of the lamina in the same position relative to one another, we may fairly assume (1) that these forces act between pairs of particles; and (2) that the action and reaction between any pair of particles consists of a pair of equal and opposite forces, such as T_1 and T_2 in the figure. Moreover, by the definition of a rigid body, the distance between any two particles remains constant when the body is displaced (§ 49). Hence the arguments of § 213 apply step for step.

The theorem also applies to two particles connected by an inextensible string passing over a smooth peg (Fig. 165).

For resolved part of u along **AC**
= rate at which **A** is approaching the peg
= rate at which string is passing over peg
= rate at which **B** is leaving the peg
= resolved part of v along **DB**.

Also tension at **A** = tension at **B**.

Fig. 165.

Hence, as in § 213,

algebraic sum of rates of working of tensions at **A** and **B** $= 0$.

Again, in the case of a particle on a fixed smooth surface, if the virtual velocity be taken along the surface, the reaction of the surface does no work, being at right angles to the displacement.

215. Recapitulation.—The connections of the various particles in a system with one another, and the restraints imposed by strings, or contact with smooth planes, etc., are usually spoken of as the Geometrical Conditions of the System.

Thus we may sum up our results in the following manner :—

If in any system under the action of forces we assign arbitrary virtual velocities to the various parts, then, when passing through the position of equilibrium, the algebraic sum of the rates of working of all the forces external and internal = 0.

But, if the virtual velocities are such that they do not violate the geometrical conditions of the system, then all internal forces, and all reactions at fixed smooth surfaces, may be omitted from the Equation of Virtual Velocities.

This equation is expressed by the formula

$$\Sigma(P u \cos\theta) = 0,$$

Σ *denoting a summation which includes all external forces with the exception of the reactions of fixed smooth surfaces.*

216. **Work of a couple.**—*If a rigid body acted on by a couple of moment M perform a complete rotation round any point (the couple maintaining the same position relative to the body), the work done by the couple* = $2\pi M$.

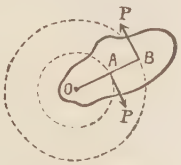


Fig. 166.

Let the rigid body rotate round **O** ; and let **OAB**, perpendicular to the lines of action of the forces, cut them in **A** and **B**, respectively.

Then suppose the forces P and P applied at **A** and **B**. **B** describes a circle of radius **OB**, and P always acts along the tangent to that circle ;

$\therefore$ work done by P at **B** in one revolution $= P \times 2\pi \,.\, \mathbf{OB}$.

Similarly, work done by P at **A** in one revolution $= -\, P \times 2\pi \,.\, \mathbf{OA}$;

$\therefore$ work done by couple $= P \,.\, 2\pi(\mathbf{OB} - \mathbf{OA}) = 2\pi \,.\, P \,.\, \mathbf{AB} \,.= 2\pi \mathbf{M}$.

The student should verify for himself the case where **O** lies between **A** and **B**.

217. We will now work some examples illustrating the application of the Principle of Virtual Velocities.

Examples.—(1) In the accompanying figure **AB** represents a plane inclined at an angle of 30° to the horizon ; **A** and **D** are smooth pegs, over which pass strings connecting the weights P, Q, and R.

Given $P = Q = \sqrt{3}$ lb., and $R = 1$ lb., find the angle which the string **D**Q makes with the inclined plane in the position of equilibrium.

Let θ be the required angle. Give the system a virtual motion *consistent with its geometrical conditions* ; suppose P moving vertically up with velocity u, and therefore Q moving down the plane with the same velocity ; and R moving vertically down with velocity v.

Then from the geometrical conditions we obtain (cf. § 214) resolved part of Q's velocity along Q**D** = resolved part of R's velocity along **D**R,

i.e., $$u \cos\theta = v \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Also, algebraic sum of rates of working of the external forces (excluding reactions at smooth surfaces) = 0 (cf. § 215) ;

i.e. algebraic sum of rates of working of the weights of P, Q, and R = 0 ;

i.e., $$P \,.\, (-u) + Q \,.\, (u \cos 60°) + R \,.\, v = 0 \quad \text{(cf. § 211)} ;$$

whence, putting in numerical values,

$$-u\sqrt{3} + \frac{u\sqrt{3}}{2} + v = 0 \ldots (2) ;$$

$\therefore$ from equation (1)

$$-u\sqrt{3} + \frac{u\sqrt{3}}{2} + u\cos\theta = 0 ;$$

whence $$\cos\theta = \frac{\sqrt{3}}{2} ;$$

$$\therefore \quad \theta = 30°,$$

or the string Q**D** is horizontal.

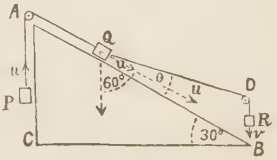


Fig. 167.

(2) A rod whose centre of gravity is not at its middle point is resting entirely within a smooth hemispherical bowl. Prove that in the position of equilibrium the centre of gravity will be vertically below the centre of the bowl.

Suppose the rod to slide inside the bowl, and let **AB**, **A′B′** represent two positions of the rod, **G** and **G** being the corresponding positions of the centre of gravity, **E** the centre of the bowl.

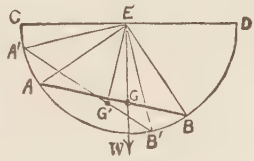


Fig. 168.

Then *the centre of gravity will always be at the same distance from* **E**.

For $\triangle AEB \equiv \triangle A'EB'$ (Euc. I. 8);

hence $\triangle EAG \equiv \triangle EA'G'$ (Euc. I. 4),

i.e. $EG = EG$;

which proves that **G** is always at the same distance from **E**.

Hence the locus of **G** is a circle; therefore its virtual velocity is always at right angles to **EG**.

Again, since the geometrical conditions are not violated, *the only force which enters into the Equation of Virtual Velocities is W*, the weight of the rod. (Cf. § 215.)

Hence, in the position of equilibrium, rate of working of $W = 0$.

Hence *the virtual velocity of* **G** *must be horizontal*; but virtual velocity of **G** is always perpendicular to **EG**;

$\therefore$ **EG** must be vertical.

(3) **ABCD** is a square framework of equal uniform rods, each of weight W, suspended from the point **A**, and kept in shape by a light string **AC**. Calculate the tension of **AC**.

This tension is one of the internal forces, and as such would not appear in the Equation of Virtual Velocities. But if T is the tension we could suppose the string removed and equilibrium maintained by an external vertical force T applied at **C**. This external force will now appear in the Equation of Virtual Velocities provided that the virtual motion is one that would be inconsistent with the action of the string **AC**.*

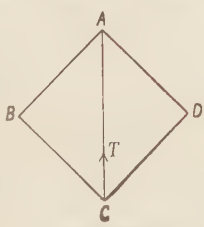


Fig. 168*a*.

Let us suppose that **A** is fixed, and **C** is travelling vertically downwards through this equilibrium position with velocity u; and let a be the length of each rod.

* Note the general rule :—In order to bring any stress into the Equation of Virtual Velocities, we must assume a motion inconsistent with the constraint which produces that stress.

Now in all positions the centre of gravity of the framework bisects **AC**. Hence the virtual velocity of the centre of gravity is $\frac{1}{2}u$ vertically downwards; thus since $4W$ is the weight of the framework, the rate of working of this weight is $4W \times \frac{1}{2}u$, *i.e.* $2Wu$. (Compare § 175).

Again, since the motion of C is directly opposed to the force T, the rate of working of T is $-Tu$. Also the external supporting force at **A** does no work, as **A** has no virtual velocity.

Thus the Equation of Virtual Velocities gives

$$2Wu - Tu = 0,$$

whence

$$T = 2W.$$

(4) **AC** is a uniform rod of weight W, freely hinged at **A** to a light rod **AB** of equal length. The system is standing on a smooth horizontal plane, and a light string **DE** connects the middle points of the rods. If the angle at **A** is 60°, determine the tension of the string.

Suppose the string removed and external forces, equal to its tension, applied at **D** and **E**. Give **A** a virtual velocity u vertically downwards.*

The forces that will enter into the Equation of Virtual Velocities are T at **D**, T at **E**, and W at **E**; P and Q the reactions of the smooth plane will not enter. It is necessary to determine the virtual velocities of **D** and **E**.

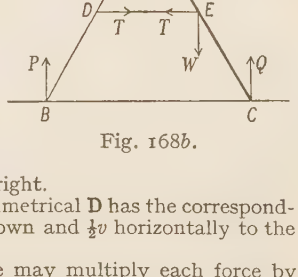


Fig. 168*b*.

Let v be the virtual velocity of **C** along the plane. Then **E** is always half as far above **BC** as **A**; hence the vertical velocity of **E** is half that of **A**, *i.e.* $\frac{1}{2}u$ vertically down. Also the distance of **E** from the vertical through **A** (which is a fixed line) is always half that of **C** from this vertical; hence the horizontal velocity of **E** is half that of **C**, *i.e.* $\frac{1}{2}v$ to the right.

Since the figure is geometrically symmetrical **D** has the corresponding velocities to **E**, viz. $\frac{1}{2}u$ vertically down and $\frac{1}{2}v$ horizontally to the left.

To obtain the rates of working we may multiply each force by the resolute (in the direction of the force) of the velocity of its point of application; for the formula for the rate of working may be written "$P \times u \cos\theta$." Thus we obtain the equation

$$T \times (-\tfrac{1}{2}v) + T \times (-\tfrac{1}{2}v) + W \times \tfrac{1}{2}u = 0,$$

or

$$W \,.\, u - 2T \,.\, v = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i.)}$$

Now we may determine the ratio of u to v by the theorem of § 212. This gives

$$u \,.\, \cos 30° = v \,.\, \cos 60°$$

or

$$u\sqrt{3} = v \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(ii.)}$$

* In point of fact if the string were cut A would not fall vertically, but this has nothing to do with the question.

Thus from (i) and (ii) we have $W \cdot u - 2T \cdot u\sqrt{3} = 0$,
whence $T = \frac{1}{6}W\sqrt{3}$.

217*a*. By means of § 215 we can give a logical proof of § 51.

Let a body be acted upon by two forces P at **A** and Q at **B** (Fig. 168*c*) ; required the conditions of equilibrium.

(i) If we give the whole body a virtual velocity at right angles to P, no work is done by P ; hence, for equilibrium no work must be done by Q. Hence Q is either zero or perpendicular to the virtual velocity, *i.e.* parallel to P.

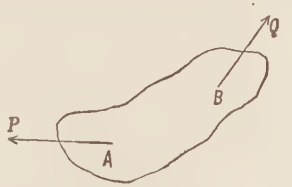


Fig. 168*c*.

(ii) If we give the whole body a virtual velocity u in the direction of P, P's rate of working is Pu ; hence, for equilibrium, Q's rate of working must be $-Pu$. Thus Q cannot be zero ; it is therefore parallel to P by (i), and its rate of working will be $-Pu$ as required if it is equal and opposite to P.

(iii) If we rotate the body about **A**, P's rate of working is zero. Hence Q's rate of working must also be zero, *i.e.* in the equilibrium position **B**'s velocity must be perpendicular to Q, and therefore to P. But **B**'s velocity is perpendicular to **AB**. Thus **AB** must coincide with P's line of action, and § 51 is completely proved.

The student may suspect that in using § 215 to prove § 51 we are arguing in a circle. But the proof of § 215 requires only (i) the conditions of equilibrium of forces acting at a point, (ii) Newton's Third Law, and (iii) certain theorems on the motion of rigidly connected points, and is therefore independent of any theorems based on § 51.

217*b*. The following proof of the Principle of Virtual Velocities for a set of coplanar forces acting on a rigid body is of some interest.

We have to prove that if a rigid lamina is under the action of a system of coplanar forces in equilibrium, and if any "virtual" motion is assigned to the lamina in its own plane consistent with its rigidity, then the algebraic sum of the rates of working of the forces is zero.

By a well known theorem in Dynamics, any motion of a rigid lamina in its own plane can be regarded at any instant as a rotation about some point in the plane, called the "instantaneous centre of rotation" (see *Tutorial Dynamics*). Let **O** be the instantaneous centre of rotation, and ω the instantaneous angular velocity of the lamina about **O** (Fig. 168*d*).

Let P be one of the forces and **A** its point of application. Then the virtual velocity of **A** is ω . **OA** at right angles to **OA**. Also if θ be the inclination of this velocity to the line of action of P, and **ON** the perpendicular to this line of action, it is easy to prove that $\theta =$ **AON**.

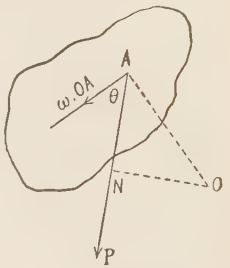


Fig. 168*d*.

Thus rate of working of $P = P$. ω . **OA**.cos $\theta = \omega$. P . **OA** cos **AON** $= \omega$. P . **ON** $= \omega \times$ moment of P about **O**.

Similarly, the rate of working of each force in the system is $\omega \times$ moment of the force about **O**. Hence the algebraic sum of the rates of working of the forces $= \omega \times$ algebraic sum of the moments of the forces about **O**.

But the system is in equilibrium by hypothesis; thus the algebraic sum of moments about **O** is zero. Hence the algebraic sum of the rates of working is also zero.

Summary of Results.

If a *particle* under the action of a system of forces in equilibrium receive a virtual displacement, the algebraic sum of the virtual works of the system is zero. (§ 208.)

If a *system of bodies* under the action of a system of forces be moving with given virtual velocities, then, when passing through the position of equilibrium, the algebraic sum of the rates of working of all the forces is zero.

If the virtual velocities do not violate the *geometrical conditions* of the system, the algebraic sum of the rates of working of the *external* forces alone (omitting reactions at smooth surfaces, and tensions of inextensible strings) is zero. (§ 215.)

Logical proof of the conditions of equilibrium of two forces acting on a rigid body. (§ 217*a*.)

EXAMPLES XVI.

(*Selected Problems* :—2, 3, 4, 6, 7, 9, 11, 13, 15 ; also Ex. XV., 9, 14, 19, 20, 25, by Virtual Velocities.)

1. Prove the results of § 44 by the Principle of Virtual Work.

2. Prove the results of § 101 by the Principle of Virtual Velocities. (Make the lever rotate round the fulcrum.)

3. A mass of $\sqrt{3}$ lb. rests on a smooth plane inclined at an angle of 30° to the horizon. It is attached to a string which passes over a smooth pulley *above* the top of the plane, and supports a mass of 1 lb. hanging vertically. Find the angle between the plane and the string by the Principle of Virtual Work.

4. Prove that the Principle of Virtual Work holds good in the first system of pulleys if there are three moveable *heavy* pulleys.

5. A weight W, constrained to move in the circumference of a vertical circle, is attached to another weight P, hanging vertically by means of a cord passing over a fixed pulley at a given point on the circle. Find the position of equilibrium of W.

6. A step-ladder of the form of the letter **A**, with both its legs inclined at an angle α to the vertical, is placed on a horizontal floor, and is held up by a cord connecting the middle points of its legs, there being no friction anywhere. Prove that, when a weight W is placed on the top of the ladder, the tension of this cord is increased by $W \tan \alpha$.

7. In the last question, prove that, when the weight W is placed on one of the steps at a height from the floor equal to $1/n$ of the height of the ladder, the increase of tension is

$$(W \tan \alpha)/n.$$

8. A square is formed of equal heavy rods of weight W. It is suspended from a peg at one corner, and kept in shape by a string joining this corner to the opposite one. Find, by Virtual Work, the tension of the string.

9. Three rigid rods **BC, CA, AB** form an equilateral triangle in a horizontal plane and have at their middle points three small smooth rings through which passes an endless string in tension T. Prove that the stress at **A** is equal to T.

10. In the preceding question, find the stress at **A** if **ABC** is a right-angled isosceles triangle, the angle at **A** being the right angle.

11. Four equal uniform rods are freely jointed together so as to form a rhombus, which is suspended by one angular point. Find the stress along a light rod joining the middle points of the two upper rods so as to prevent the rhombus collapsing ; given that the weight of the rhombus $= W$, and the angle from which it is suspended is 2α.

12. A regular octagon **ABCDEFGH** is formed of eight equal heavy rods jointed together ; the rod **AB** is fixed horizontally and the framework hangs from it, the regular octagonal form being maintained by two weightless rods joining **C** to **H** and **D** to **G**. Show that the reactions along these rods are $3W$ and W respectively, where W is the weight of each side of the octagon.

13. A framework of four equal rods freely jointed together lies on a horizontal plane in the form of a square ; the middle points of contiguous sides are connected by strings in which the tensions are T_1, T_2, T_3, T_4 respectively in order round the square : prove that for equilibrium it is necessary that

$$T_1 + T_3 = T_2 + T_4.$$

14. A parallelogram **ABCD** is formed of four uniform rods freely jointed at the corners. **AB** is fixed horizontally and **A** is attached to the opposite joint **C** by a string of length l. **AC** is the shorter diagonal, and α the acute angle of the parallelogram. Show that the tension of the string is $Wl \cot \alpha/2a$, where W is the weight of the four rods and a the length of the side **AB**.

15. A smooth horizontal rail is parallel to a smooth wall and one foot from it ; a uniform beam 16 feet long rests over the rail with one end pressing against the wall. Find by Virtual Work (or Virtual Velocities) the angle between the beam and the wall in the position of equilibrium.

EXAMINATION PAPER VII.

1. State and prove the conditions of equilibrium of any number of forces in a plane.

2. A system of forces in a plane is such that the sum of their moments about a point **A** in the plane vanishes. Prove that the forces are either in equilibrium or have a resultant which passes through **A**.

3. Prove that, if n forces $P_1, P_2, \ldots P_n$ act in a plane on a rigid body, if rectangular axes **Ox, Oy** be taken in that plane, if the components of any force such as P_r parallel to those axes be X_r, Y_r, and if the coordinates of the point of application of P_r be x_r, y_r, then the system $P_1, P_2, \ldots P_n$ is equivalent to a force ΣX_r acting along the axis of x, a force ΣY_r acting along the axis of y, and a couple $\Sigma(x_r Y_r - y_r X_r)$ in the plane of x, y; Σ denoting summation for all values of the suffix r from 1 to n.

4. A frame, consisting of two pairs of parallel bars hinged without friction at their four intersections, being supposed in strained equilibrium under the tensions of two diagonal ties, determine, given all particulars, the ratio of the tensions in the ties.

5. A thin board is free to move on a horizontal plane, and is acted on by a system of forces which all lie in that plane. Prove that there is usually a line in the plane such that, if it is rigidly connected with the board and any point on it is held fixed, the board will not move. What is the exceptional case?

6. A rough heavy body, bounded by a curved surface, rests upon two others which themselves rest on a rough horizontal plane. Prove that the three centres of gravity and the four points of contact lie in one plane.

7. State the Principle of Virtual Velocities, and apply it to find the relation between the power and weight in the third system of pulleys, the strings being parallel, and the number of pulleys n.

[The third system of pulleys is that in which each string is attached at one end to the weight.]

8. Describe the action of the oar of a boat, and compare the velocity of the boat and the velocity of the hands pulling the oar. Hence, find the ratio of the resistance to the effort if the oar be l feet long and the rowlock a feet from the hands.

9. Two weightless strings are tied to the ends of a uniform bar; each passes over a pulley, and has a weight attached to it equal to the weight of the bar. Prove that, when in equilibrium, the bar is horizontal, and find the inclination of the strings to the vertical.

10. A uniform bar **AB**, of weight W, rests in a horizontal position on two supports at the ends, and an ideal section is made at any point **P**. Determine, for the part **BP**, the force at **P**, and the couple, which might statically replace the supporting action of the portion **AP**.

CHAPTER XVII.

THE LAWS OF FRICTION.

218. We have often had occasion to make use of the principle that when two perfectly smooth bodies rest against one another the reaction against them is perpendicular, or, as it is sometimes called, "**normal**," to the surfaces in contact. But no body can be made perfectly smooth, however much its surface may be polished or smoothed down. Hence all bodies are capable of exerting a certain resistance which tends to prevent them from sliding along one another. This resistance is called **friction.**

The laws of friction have to be found by experiment. Accordingly, we shall begin by describing certain experiments which might be used to prove them.

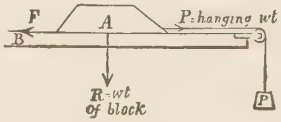


Fig. 169.

219. **Experiments on a horizontal plane.—I.** Let a block of any substance (whose faces are not all equal) **A** be placed on a horizontal table or surface of the same or any other substance, and let the block be tied to a string passing over a small pulley at the edge of the table, and carrying at its other end a light scale pan, by which weights P can be placed tending to pull the block along the plane (Fig. 169).

Then it will be found that, unless the total suspended weight at **P** exceeds a certain amount, the block will not slide along the plane. Since the block remains in equilibrium, the table must exert on it a horizontal force equal and opposite to the pull of the string, and therefore equal to the weight attached at **P**.

By increasing the weights at **P**, we shall at last reach a limit beyond which it is impossible to add further weights without making the block slide. After this limit has been passed, the friction is no longer able to counteract the pull of the string, and the block begins to slide.

DEFINITION.—The greatest amount of friction that can be called into play between two surfaces is called the **limiting friction.** Hence we have the following :—

LAW I.—**The amount of friction called into play between two surfaces in contact will just suffice to prevent them from sliding on one another, provided that this amount does not exceed the limiting friction.**

220. **Experiment II.**—Let weights be placed at **P** until the addition of any more weights would cause the block to slide. Then these weights are exactly equal to the limiting friction. Now double the load **A** resting on the table by placing another equal block of the same substance on top of the first, or otherwise. It will be found that the weights at **P** will have to be doubled before the block begins to move. If the weight of **A** be trebled, the weights at **P** will have to be trebled before the block slides, and so on.

Hence the limiting friction is proportional to the weight of **A**, say R. But the plane exerts an equal and opposite upward reaction R, preventing the weight **A** from penetrating into it. This reaction is perpendicular to the plane, and is therefore the normal thrust between the surfaces. Hence we have

LAW II.—**The limiting friction is proportional to the normal thrust between the surfaces in contact.**

221. **Experiment III.**—Let the weights at **P** be again adjusted till the block is on the point of moving. Let the block be turned over on one of its other faces. Then, with the same weights at **P**, the block will still be on the point of moving. This shows that the limiting friction is the same as before, although a different surface is in contact with the table. Hence

LAW III.—**The limiting friction does not depend on the areas of the surfaces of the two bodies.**

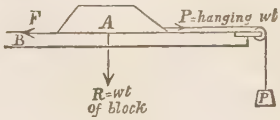


Fig. 169.

222. **Experiment IV.**—Let the weights at **P** be sufficient to draw **A** along the plane. Then careful experiments prove that **A** and **P** move with approximately uniform acceleration, thus showing that they are acted on by a force that is approximately constant. The only forces on them are the weight at **P** and the friction on **A**. Hence the latter is approximately constant; and, since the velocity of **A** is continually increasing, we see that

LAW IV.—**When slipping takes place, the friction is approximately independent of the velocities of the surfaces in contact.**

Careful experiments show that the friction is not quite constant, but varies slightly as the velocity changes.

223. **Experiment V.**—If bodies of different material, but of the same weights, be substituted for **A**, the weights at **P** required to set them in motion will be different. They will also be altered by altering the material of which the surface of the table is formed.

Hence **the friction between two bodies varies according to the nature of the material forming their surfaces.**

224. **Coefficient of friction.**—DEFINITION.—The ratio of the limiting friction to the normal thrust is called the **coefficient of friction.** It is generally denoted by the Greek letter μ (mu). Thus, if R denote the normal thrust and F the limiting friction,

$$\frac{F}{R} = \mu, \text{ or } F = \mu R \quad \ldots\ldots\ldots \quad (1).$$

Law II. shows that μ is the same for different weights formed of the same material. Law III. shows that it is independent of the area of the surfaces in contact, and Experiment V. shows that μ varies with the nature of the materials forming these surfaces.

Even for the same materials, the coefficient of friction can be diminished by smoothing their surfaces, or increased by roughening them up with coarse sand-paper. By greasing or oiling the surfaces, the coefficient of friction can be reduced considerably; this is the reason why in machinery all sliding surfaces are **lubricated** (as it is called) by oiling.

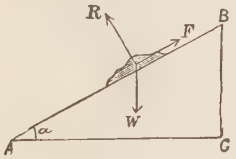


Fig. 170.

225. **To determine the coefficient of friction by experiments with inclined planes.—Experiment VI.**—Let a body W of any substance, and of such a shape as not easily to topple over when tilted on one side, be placed on a flat slab of any material, and let the slab be gradually tilted up on one side so as to form an inclined plane. The body will at first continue to rest on the plane, being kept from sliding down by the friction of the plane. When the plane has reached a certain inclination, the body will just be on the point of sliding, and any further increase in the inclination will cause the body to slip down. At this particular inclination the friction is limiting.

Now the body is acted on by its weight W (Fig. 170) the normal reaction R (say) of the plane and the force of friction F (say) acting along the plane and preventing the body from sliding down. The conditions of equilibrium are therefore the same as if the inclined plane were smooth and the body were supported by an effort F applied along the plane. Hence

$$\frac{F}{\mathbf{BC}} = \frac{W}{\mathbf{AB}} = \frac{R}{\mathbf{CA}},$$

or $$\frac{F}{\text{height of plane}} = \frac{W}{\text{length of plane}} = \frac{R}{\text{base of plane}};$$

$$\therefore \text{ coefficient of friction } \mu = \frac{F}{R} = \frac{\mathbf{BC}}{\mathbf{CA}} = \frac{\text{height of plane}}{\text{base of plane}}.$$

Hence, by measuring the height and the base of the plane, the coefficient of friction may be found.

COROLLARY.—Hence

μ = tan **BAC** = tangent of inclination of the plane.... (2).

226. **Experiment VII.**—The laws of friction can also be verified by means of an inclined plane. For, if different bodies of the same substance be placed on the plane, and the plane gradually tilted more and more, the bodies will all begin to slide down together, and when they slide they will all slide with the same approximately uniform acceleration. This shows that the coefficient of friction is the same for any two bodies whose surfaces are of the same substance, that it is independent of the areas of the surfaces in contact, and that it is approximately independent of the velocity when the bodies slide.

227. Angle of friction.—Total reaction.

DEFINITION 1.—The angle whose tangent is μ, the coefficient of friction, is called the **angle of friction.** This angle is expressed by the trigonometrical notation

$$\text{angle of friction } \theta = \tan^{-1}\mu \ldots\ldots\ldots \quad (2a).$$

[This may be regarded as equivalent to the statement

$$\tan\theta = \ldots\ldots\ldots\ldots\ldots\ldots (2).]$$

DEFINITION 2.—The resultant force which two rough bodies exert on each other (whether friction is limiting or not) is called the **total reaction** between the bodies.

Thus the total reaction is the resultant of R, the normal reaction and F, the force of friction.

DEFINITION 3.—A body is said to be in **limiting equilibrium** when the limiting friction or greatest possible friction has to be called into play to prevent it from slipping.

From § 225 (2) we now see that *the greatest inclination at which a body will remain at rest on an inclined plane is equal to the angle of friction.*

Let us consider the case of equilibrium on an inclined plane a little more closely. If P be the total reaction of the plane, the conditions of equilibrium require that P

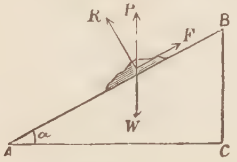


Fig. 171.

should be equal and opposite to the weight W (Fig. 171), since P, W are the only forces on the body. Hence P must be vertical, and therefore

$$\angle \text{ between } P \text{ and } R = \angle \mathbf{BAC}.$$

But, when the body is on the point of slipping, we have seen that $\angle$ **BAC** is equal to the angle of friction.

Thus, *in limiting equilibrium the total reaction makes with the normal an angle equal to the angle of friction.*

If the inclination is *less* than the angle of friction, no slipping will take place. If *greater*, the body will slip down the plane.

Hence we have an easy way of finding out whether two bodies will slip when placed in contact under any forces. We only have to construct the direction of the

reaction at the point of contact which would be required to keep them in equilibrium. The bodies will slip or stick fast according as the angle which the reaction makes with the normal to the surface is an angle greater or less than the angle of friction.

228. **Limiting equilibrium of bodies of any shape.**—We have considered above the case of a body on an inclined plane, but the same is true for any bodies in contact.

Thus, let **V, W** be two bodies touching each other at **A** in the plane **AX**. Then the line **BAC** drawn through **A** perpendicular to **AX** is the **common normal** at **A**, and the components of the total reaction at **A** resolved along **AB** and in the plane **AX** are the normal reaction and friction, respectively.

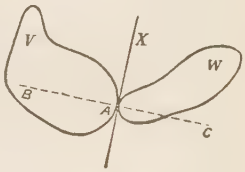


Fig. 172.

Calling these R and F, their resultant (the total reaction) makes with **AB** an angle whose tangent is F/R (*vide* § 24, p. 17). For equilibrium, F must be less than μR or $F/R < \mu$; hence the inclination of the total reaction to the normal $< \tan^{-1}\mu$, *i.e.*, < angle of friction.

Examples.—(1) A ladder is resting on a rough ground, and leaning against a rough wall; find the greatest angle at which the ladder can be inclined to the vertical without slipping; given the coefficients of friction for the ground and wall are $\frac{4}{27}\sqrt{3}$ and $\frac{1}{4}\sqrt{3}$, respectively.

Let **AB** represent the ladder, **BC** the ground, and **CA** the wall, **DE** the line of action of the weight, R and S the normal reactions at **B** and **A**, respectively, and l the length of the ladder (Fig. 173). Let α be the greatest angle at which the ladder can be inclined to the wall. Then the limiting frictions act at **A** and **B**, as represented in the figure.

Now apply § 201, Theorem (2).

Resolving vertically, $R + \frac{S\sqrt{3}}{4} - W = 0$.. (i.),

,, horizontally, $S - \frac{4R\sqrt{3}}{27} = 0$.. (ii.).

Taking moments about **C**,

$$S \,.\, \mathbf{AC} + W \,.\, \mathbf{EC} - R \,.\, \mathbf{BC} = 0;$$

i.e., $S \,.\, l\cos\alpha + W \,.\, \frac{1}{2}l\sin\alpha - R \,.\, l\sin\alpha = 0$.. (iii.).

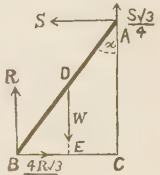


Fig. 173.

From (i.) and (ii.) we obtain

$$R = \frac{9S\sqrt{3}}{4} \quad \text{and} \quad W = \frac{5S\sqrt{3}}{2}.$$

Substituting in (iii.), dividing through by $Sl \cos \alpha$, and simplifying, we find $\tan \alpha = \frac{1}{\sqrt{3}}$, whence $\alpha = 30°$.

(2) A rectangular block, a ft. long and b ft. high, is at rest on a rough horizontal plane. A string is attached to the top of the block, and pulled horizontally with increasing force P. Required to determine whether the block will tilt over before it slides, or *vice versa* (Fig. 174).

We must first determine what force will be required to make the block tilt, supposing it does not slide. If it is on the point of tilting,

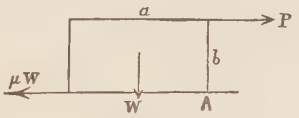


Fig. 174.

i.e., of rotating about **A**, the moments of P and W about **A** must balance, *i.e.*, $Pb = W \cdot \frac{1}{2}a$, whence $P = Wa/2b$.

Next, to determine what force will make the block slide, supposing it does not tilt.

The only forces with a horizontal component are P and the friction; but the friction cannot exceed μW. Hence the block will be on the point of sliding when $P = \mu W$.

Thus the block will slide first if $\mu W < Wa/2b$, *i.e.*, if $\mu < a/2b$.

229. **Loss of work by friction.**—When a body is drawn along a rough plane, the friction always tends to retard it, and therefore work has to be done against friction in addition to any other work that is done against other forces that may be present. Hence, when slipping takes place there is a loss of work due to friction.

The action and reaction of two bodies due to friction are equal and opposite, but the points of the bodies in contact do not move together when slipping takes place; hence the works done are not equal and opposite (§ 213).

Friction always tends to retard any relative slipping motion of the two surfaces between which it acts. It never tends to produce slipping. Hence any work that has been done against friction can never be recovered, at any rate, in the form of useful mechanical work, but for all practical purposes such work must be regarded as lost.

It is, in reality, mostly converted into heat as a rule. In a frictional electric machine, part of the work done against friction takes the form of electrical energy. The Principle of Conservation of Energy tells us that the lost energy always appears in *some* form or other.

230. To find the least force which will pull a body up a rough inclined plane, the force being supposed to act parallel to the plane.

By the least force is meant that force which will draw the body along with uniform velocity—*i.e., without acceleration*. Such a force will just balance the limiting friction and the resolved part of the weight which it has to overcome; any *greater* force will draw the body up with constant acceleration.

Suppose the body O to be just on the point of moving up the inclined plane AC under the action of a force P (Fig. 175). Then the

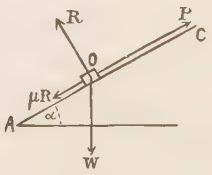


Fig. 175.

limiting friction μR will be acting down the plane. Resolving at right angles to the plane, we find $R = W \cos \alpha$.

Resolving parallel to the plane, we find

$$P = \mu R + W \sin \alpha = \mu W \cos \alpha + W \sin \alpha \quad \ldots\ldots\ldots (3).$$

In the same way it may be proved that, if the inclination of the plane is less than the angle of friction, the least force required to make it move down the plane will be $W \cos \alpha - W \sin \alpha$; and that, if the inclination of the plane is greater than the angle of friction, the least force which will prevent it moving down the plane is

$$W \sin \alpha - \mu W \cos \alpha.$$

231. Equilibrium of the screw press, taking account of friction.

To determine approximately the relation between the effort P and the resistance Q in a screw press, with rectangular thread of small breadth, if the coefficient of friction between the thread of the screw and the groove in which it runs is μ.

[Fig. 176 represents the screw, and Fig. 177 represents a section through the axis showing the rectangular form of the thread, a screw with triangular thread being shown for the purpose of comparison in Fig. 178.]

Let the radius of the cylinder of the screw be r; let **OX** be the arm, of length a, at the end of which the force P is applied. Imagine the cylinder of the screw to be hollow, and to be cut down the line **AE**, and spread out into a rectangle, as in Fig. 179. Then the line representing the thread of the screw would be cut through at the points **B, C, D, ..**, and would appear as the parallel straight lines $\mathbf{A_2B_1}$, $\mathbf{B_2C_1}$, $\mathbf{C_2D_1}$, ... Also $\mathbf{B_1B_2}$, $\mathbf{C_1C_2}$, .. would be perpendicular to $\mathbf{A_1E_1}$. Let $\angle \mathbf{A_2B\,B_2} = \alpha$; then α represents the inclination of the thread of the screw to the horizontal (the screw being supposed vertical).

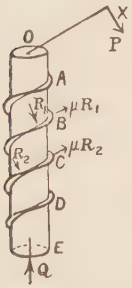
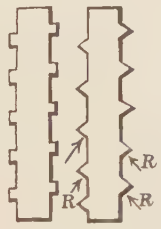


Fig. 176.



Figs. 177, 178.

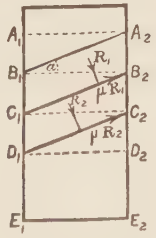


Fig. 179.

Then the forces acting on the screw are

(1) P, applied horizontally at the end of the arm OX;

(2) Q, the resistance;

(3) a series of normal reactions R_1, R_2, .. at the various points of contact of the thread of the screw with the groove in which it works;

(4) the corresponding series of limiting frictions μR_1, μR_2, ...

Since the breadth of the thread is assumed to be small, the points of application of the reactions between the thread and the groove may be treated as lying approximately on the surface of the cylinder.

Also, since the thread of the screw is inclined at an angle α to the horizontal and is of rectangular section, each normal reaction will be inclined at an angle α to the vertical, and will be, approximately, in the tangent plane to the cylinder at its point of application.* Also each corresponding force of friction will for the same reason lie approximately in the same tangent plane, and make an angle α with the horizon.

Thus R_1 can be resolved into $R_1 \cos \alpha$ vertically downwards, and $R_1 \sin \alpha$ horizontal and tangential to the cylinder; similarly for R_2, ..

Also μR_1 can be resolved into $\mu R_1 \sin \alpha$ vertically upwards, and $\mu R_1 \cos \alpha$ horizontal and tangential to the cylinder.

Thus, resolving vertically, we obtain

$$Q = R_1 \cos \alpha + R_2 \cos \alpha + \ldots - \mu R_1 \sin \alpha - \mu R_2 \sin \alpha - \ldots$$
$$= (\cos \alpha - \mu \sin \alpha)(R_1 + R_2 + \ldots),$$

i.e., $Q = (\cos \alpha - \mu \sin \alpha) \Sigma (R)$ ..... (i.),

where Σ denotes summation.

Again, taking moments about the axis of the screw, we notice that Q and the vertical resolved forces do not tend to turn the screw; while the horizontal resolved forces, being regarded as tangential to the cylinder, are at a perpendicular distance r (approximately) from the axis.

Thus $Pa = rR_1 \sin \alpha + rR_2 \sin \alpha + \ldots + r\mu R_1 \cos \alpha + r\mu R_2 \cos \alpha + \ldots$
$= r (\sin \alpha + \mu \cos \alpha) \Sigma (R)$ ..... (ii.).

From equations (i.) and (ii.), we find

$$P = \frac{r}{a} \frac{\sin \alpha + \mu \cos \alpha}{\cos \alpha - \mu \sin \alpha} Q.$$

But, if θ be the angle of friction $\mu = \tan \theta$; thus

$$P = \frac{r}{a} \frac{\sin \alpha + \cos \alpha \tan \theta}{\cos \alpha - \sin \alpha \tan \theta} Q = \frac{r}{a} \frac{\sin \alpha \cos \theta + \cos \alpha \sin \theta}{\cos \alpha \cos \theta - \sin \alpha \sin \theta} Q$$

$$= \frac{r}{a} \frac{\sin (\alpha + \theta)}{\cos (\alpha + \theta)} Q = \frac{r}{a} \cdot \tan (\alpha + \theta) Q \quad \ldots\ldots\ (4).$$

If b denotes the step or distance between two threads (B_1, C_1),

$$\tan \alpha = \frac{b}{2\pi r} \quad \ldots\ldots\ (5),$$

whence α may be found.

Cor. I.—If the screw is smooth, $\theta = 0$;

$$\therefore \quad P = \frac{r}{a} \tan \alpha \, Q = \frac{Ql}{2\pi l} \qquad \text{[from (5)]},$$

agreeing with the result of § 132.

---

* If the thread were *triangular*, the reactions R would act as in Fig. 177, and would not be tangential to the cylinder

Cor. II.—Comparing the last two results, we see that the effect of friction is equivalent to increasing the inclination of the thread to the horizon by the angle of friction.

Cor. III.—Substituting in (4) the value of tan α from (5), and μ for tan θ, we obtain

$$P = \frac{r}{a}\frac{b/2\pi r + \mu}{1 - \mu b/2\pi r}Q = \frac{r}{a}\frac{b + 2\pi r\mu}{2\pi r - b\mu} \quad \ldots\ldots\ldots \quad (4a).$$

232. In like manner we can consider the case in which the screw is on the point of untwisting, by supposing that the friction acts up the screw. In this case, if $\alpha > \theta$, the force required to prevent the press from unscrewing itself under the action of Q will be given by

$$P = r \tan (\alpha - \theta)\, Q/a.$$

If $\alpha < \theta$ (as is usually the case in practice), the force required to unscrew the press will be given by

$$P = r \tan (\theta - \alpha)\, Q/a.$$

Summary of Results.

The magnitude of the *limiting friction* depends only on the roughness of the surfaces in contact, and the normal thrust between them;

$$F = \mu R \quad \ldots\ldots\ldots\ldots\ldots \quad (1);$$

$$\mu = \tan \theta \quad \ldots\ldots\ldots\ldots\ldots \quad (2).$$

The *least force* parallel to the plane which will pull a body up a *rough inclined plane* is

$$W (\sin \alpha + \mu \cos \alpha) \quad \ldots\ldots\ldots\ldots \quad (3).$$

In the *rough screw press* with a narrow rectangular thread,

$$P = r \tan (\alpha + \theta)\, Q/a \quad \ldots\ldots\ldots\ldots \quad (4).$$

EXAMPLES XVII.

(*Selected Problems* :—1, 3, 4, 5, 6, 8, 9, 10, 11, 12, 14, 16, 18, 21, 22, 24, 25, 28, 32, 33.)

1. A bead runs on a rough circular wire whose plane is vertical; if r is the radius of the circle, and θ the angle of friction, prove that the bead can be in equilibrium at any point of a certain arc whose length is $2r\theta$.

2. The radius of a screw is 2 in., its step is 88/21 in.; the coefficient of friction between the screw and the groove in which it runs is 1/7; determine the mechanical advantage when worked by an arm 2 ft. long.

3. Determine the mechanical advantage of a screw if the radius of the screw is r in., the step b in., the arm a in., and the coefficient of friction μ.

4. Determine the least force parallel to the plane which will pull a mass of 4 lb. up a rough plane at an inclination of 30° to the horizon, if the angle of friction is 30°.

5. Determine the least force parallel to the plane which will prevent a mass of 4 lb. from sliding down a rough plane inclined to the horizon at an angle of 60°, if the angle of friction is 30°.

6. Determine the least force parallel to the plane which will pull a mass of 20 lb. down a rough plane at an inclination of 30° to the horizon, if the angle of friction is 60°.

7. If the least force parallel to the plane which will pull a weight up a rough plane inclined at an angle of 45° to the horizon is $\sqrt{2}$ times the weight, find the coefficient of friction.

8. A beam, a in. broad and b in. thick, is lying on a rough plane parallel to one edge. If the plane be gradually tilted up about this edge, prove that the beam will slide before rolling if $a > b\mu$.

9. Prove that, to move a particle resting on a rough horizontal plane with the least possible effort, the force should be applied in a direction inclined to the plane at the angle of friction.

10. Prove the same result for moving a particle up a rough inclined plane.

11. A ladder resting against a rough wall, and on a rough ground, is on the point of slipping when the distance of its foot from the wall is one-half of its length. Determine the coefficients of friction with the ground and with the wall, respectively. Given that their product is $\frac{1}{6}$.

12. A ladder on a rough ground, against a rough wall, is on the point of slipping down if the top of the ladder is four times as far from the bottom of the wall as is the bottom of the ladder. The coefficient of friction with the ground is ·12 ; determine that with the wall.

13. The two ends of a heavy uniform beam are in contact with two planes inclined at angles of 60° and 30° to the horizon, respectively. Prove that the beam can just rest horizontally, if either plane is smooth, and the angle of friction of the other is 30°.

14. A heavy uniform beam rests with one end on a rough horizontal plane, and the other on an equally rough plane inclined at 60° to the horizon. If the beam is on the point of slipping down when inclined at 30° to the horizon, show that the coefficient of friction is determined from the equation $\mu^2 + 4\mu\sqrt{3} - 3 = 0$.

15. If the beam in Question 13 can just rest horizontally when both planes are rough, prove that the sum of the two angles of friction is 30°.

16. A cylinder rests on a perfectly rough horizontal plane; leaning against it is a heavy bar, hinged to the horizontal plane at one end. If θ be the angle of friction for the bar, find the greatest angle at which the bar can be inclined to the plane.

17. A square lamina is standing upright on a rough horizontal plane. A string is attached to one of the upper corners, and pulled with slowly increasing force in a direction at right angles to the diagonal through that corner. Prove that the lamina will tilt before it slides if the coefficient of friction is greater than $\frac{1}{3}$.

18. A particle is tied to one end of a light string, the other end of which is fixed to a point in a rough plane inclined to the horizon at an angle α, which is greater than the angle of friction θ. Prove that the particle cannot remain at rest if the angle between the string and the line of greatest slope is greater* than

$$\sin^{-1}(\tan\theta \cot\alpha).$$

19. A uniform bar **AB**, of weight W, rests in a horizontal position with the end **A** against a rough wall, being supported by a string connecting its middle point with a point of the wall vertically above **A** at a distance equal to $\frac{1}{2}$**AB**. Prove that, if a weight nW be suspended from **B**, the horizontal and vertical components of the reaction of the wall at **A** will be $(2n + 1)\ W$ and nW, respectively.

20. Find the greatest and least weight by which a weight of 10 lb. can be sustained at rest on a rough plane inclined at an angle $\tan^{-1}\frac{3}{4}$ to the horizon, the coefficient of friction being $\frac{1}{4}$, and the weights being severally connected with the weight of 10 lb. by a string passing over a smooth pulley at the top of the plane.

---

* $\sin^{-1} x$ denotes the angle whose sine is x, just as $\tan^{-1}\mu$ denotes the angle whose tangent is μ. (§ 227, Def. 1.)

21. A weight of 5 lb. can just be supported on a rough inclined plane by a weight of 2 lb., or can just support a weight of 4 lb. suspended by a string passing over a smooth pulley at the top of the plane; find the coefficient of friction, and the sine of the inclination of the plane to the horizon.

22. A ladder rests against a vertical wall, the coefficients of friction with the wall and ground being μ and μ'. If both points of contact be on the point of slipping at the same time, show that

$$2 \tan \theta = 1 / \mu' - \mu,$$

θ being the inclination of the ladder to the horizontal.

23. Bodies weighing 2 lb. and 1 lb. are placed on the two equally rough slopes of a double inclined plane, whose angles of inclination to the horizon are 60° and 30° respectively, and are connected by a string passing over a small smooth pulley at the common vertex of the planes. If the heavier body is on the point of moving downwards, show that the coefficient of friction is $5\sqrt{3} - 8$.

24. Two equal weights connected by a string rest in equilibrium, one on each of two rough inclined planes of the same height and placed back to back, the string passing over a smooth pulley at the top of either plane. Show that the difference between the inclinations of the planes cannot exceed twice the angle of friction.

25. A sphere rests on a rough inclined plane, the coefficient of friction being μ, and is supported by a horizontal string attached to the highest point of the sphere. Find the inclination of the plane when the sphere is on the point of slipping.

26. A pair of steps (which may be regarded as two equal beams jointed at one extremity, and connected by a rope which is fastened to points on the beams equally distant from their common extremity) is placed on a rough horizontal plane. Find (1) when equilibrium is possible with the rope slack, (2) the tension in the rope when it is tight.

27. A square lamina, whose plane is vertical, rests with the ends of a side AB against a rough vertical wall and a rough horizontal plane respectively. Show that if the coefficients of friction are both unity, AB may make any angle less than 45° with the horizontal plane.

28. A heavy body is supported on a rough plane, inclined to the horizon at an angle α, by a force inclined to the plane at an angle θ. Find the least and greatest value of the force by which equilibrium is maintained.

29. A body is placed on a rough inclined plane. Find the direction and magnitude of (1) the least force which will make the body move up the plane, and (2) the least force which will keep the body from sliding down the plane.

30. **A** and **B** are two rough pegs on which rests a uniform rod. **AB** is inclined at an angle α to the horizon, and the coefficients of friction for **A, B,** are μ_1, μ_2, respectively. If the rod is in limiting equilibrium show that its centre of gravity divides **AB** in the ratio

$$\mu_1 - \tan\alpha : \tan\alpha - \mu_2,$$

given $\mu_1 > \tan\alpha > \mu_2$.

31. A trapezium **ABCD,** two of whose opposite sides are equal, stands with its plane vertical, so that **CD,** the longer of the two parallel sides, rests on a horizontal table, and a rough heavy string (of length $= 2AB$) rests on the top of the trapezium. Prove that the lengths of the portions of the string which rest on the equal sides **AD, BC,** when the string is on the point of slipping down one of them, are in the ratio

$$\sin(\tfrac{1}{2}\theta + \epsilon) : \sin(\tfrac{1}{2}\theta - \epsilon),$$

where θ is the angle which each of the equal sides makes with the horizon, and ϵ is the angle of friction.

32. A uniform heavy rod is placed inside a rough spherical cup ; it is in a vertical plane and subtends a right angle at the centre of the cup. Find in terms of the angle of friction, the greatest angle it can make with the horizontal.

33. A wheel of radius R and weight W is supported by a fixed axle of radius r, and carries a particle of weight w at a point on its rim. If the coefficient of friction between the wheel and axle is μ, and if the bearing is sufficiently loose that there is contact between them at one point only, show that for equilibrium the radius to the particle must not be inclined to the vertical at an angle greater than

$$\sin^{-1}\mu r(W+w)/\{wR\sqrt{(1+\mu)}\}.$$

CHAPTER XVIII.

FURTHER DETERMINATIONS OF THE CENTRE OF GRAVITY.

233. **To find the coordinates of the C.G. of a triangle** *referred to any two lines at right angles in its plane.*

Let (x_1, y_1), (x_2, y_2), (x_3, y_3) be the coordinates of the three vertices **A**, **B**, **C**, so that $\mathbf{OK} = x_1$, $\mathbf{KA} = y_1$, etc.

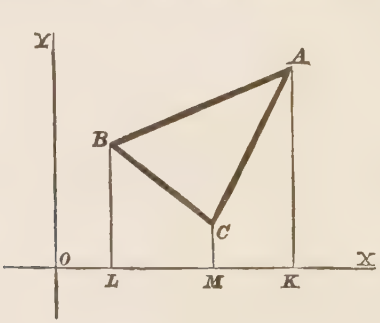


Fig. 180.

Then the required C.G. is the C.G. of three equal weights placed at **A**, **B**, **C**. Taking each of these equal weights to be unity, the formulae of § 174 give

$$\bar{x} = \frac{x_1 + x_2 + x_3}{3}, \qquad \bar{y} = \frac{y_1 + y_2 + y_3}{3}.$$

234. The formulae of § 174, quoted above, determine the centre of gravity of a number of weights or bodies whose centres of gravity are all in one plane, referring their positions to two straight lines at right angles in that plane. These formulae can be expressed in the Σ notation very briefly thus :—

$$\bar{x} = \frac{\Sigma (wx)}{\Sigma (w)}, \qquad \bar{y} = \frac{\Sigma (wy)}{\Sigma (w)}.$$

We shall now prove that similar expressions can be used to find the C.G. of a body from which part has been cut off, with only this difference, that *the weight of the portion removed has a minus sign prefixed to it* both in the numerator and denominator of the fraction.

235. To find the C.G. of a body from which part has been cut off, referred to two lines at right angles *in the same plane with the* C.G. *of the whole and that of the part.*

Let **A** be the C.G. of the whole body, W its weight, and let a portion of weight w be removed, whose C.G. before

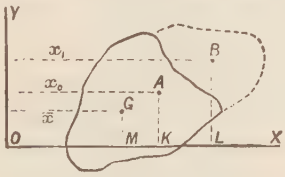


Fig. 181

removal is at **B**. It is required to find **G**, the C.G. of the remaining portion.

Let **OX**, **OY** be two straight lines at right angles in the same plane with **A**, **B**, **G**, and let the distances of **A**, **B** from **OY** be x_0, x_1. Let $\bar{x}$ be the required distance of **G** from **OY**.

Then the whole body (weight W) is made up of the portions w and $W - w$, whose centres of gravity are at **G** and **B**. Hence, if **OY** be placed vertical, the equation of moments about **O** gives, in the notation of the figure,

$$W \times \mathbf{OK} = w \times \mathbf{OL} + (W - w) \times \mathbf{OM},$$

or $Wx_0 = (W - w)\bar{x} + wx_1$, whence $(W - w)\bar{x} = Wx_0 - wx_1$;

$$\therefore \quad \bar{x} = \frac{Wx_0 - wx_1}{W - w}.$$

Again, let y_0, y_1, $\bar{y}$ be the distances of **A, B, G** from **OX** (*i.e.* the distances **KA, LB, MG**); then, by turning the body and axes round till **OX** is vertical and downwards, we obtain, in like manner,

$$\bar{y} = \frac{Wy_0 - wy_1}{W - w}.$$

Examples.—(1) From one corner of a square lamina **ABCD,** whose side is 8 in., a smaller square **EBFH** is cut off, whose side is 6 in. To find the distances of the C.G. of the remaining part from the sides **AD, DC.**

Let **K** be the C.G. of the large, and **L** that of the small, square.

The weights of the whole square and the part removed are proportional to their areas, *i.e.* as 64 : 36, or 16 : 9.

We may, therefore, denote them by $16w$ and $9w$, respectively, where w is some constant.

The distance of **K** from **AD** is 4 in.

The distance of **L** from **EH** is 3 in.; also **EH** is 2 in. distant from **AD**;

$\therefore$ the distance of **L** from **AD** is $3 + 2$ in. $= 5$ in.

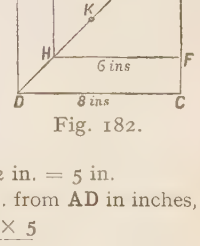


Fig. 182.

Hence, if x be the required distance of the C.G. from **AD** in inches,

$$x = \frac{16w \times 4 - 9w \times 5}{16w - 9w}$$

$$= \frac{64 - 45}{7} = \frac{19}{7}$$

The required C.G. is therefore $2\frac{5}{7}$ in. from **AD,** and, evidently (by symmetry), it is also $2\frac{5}{7}$ in. from **DC.**

(2) To apply the present method to the folded-over square of Ex. 1, p. 220.

The figure in question may be regarded as formed by taking the whole square, removing the triangle **AEF,** and adding the triangle **EOF.** If x denote the distance of the C.G. to the right of **O,** we may write x down at once by the formula

$$x = \frac{W \times \mathrm{O} - \frac{1}{8}W \times \mathrm{OK} + \frac{1}{8}W \times \mathrm{OL}}{W - \frac{1}{8}W + \frac{1}{8}W}$$

$$= -\tfrac{1}{8}\mathrm{OK} + \tfrac{1}{8}\mathrm{OL} = -\tfrac{1}{8} \cdot \tfrac{2}{5}a + \tfrac{1}{8} \cdot \tfrac{1}{3}a = -\tfrac{1}{24}a,$$

and the minus sign shows that **G** is $\frac{1}{24}a$ to the left of **O.**

236. **A frustum** of a pyramid or cone is the portion included between the base and any plane drawn parallel to it.

In Fig. 183 the lower portion of the pyramid **OABCD**, cut off by the plane **abcd**, is called a frustum of the pyramid.

Before applying the method of the last article to find the C.G. of a frustum of a pyramid or cone, it will be necessary to compare the volume of the whole pyramid **OABCD** with that of the upper pyramid **Oabcd** cut off, and we shall now prove that *these volumes are in the ratio of the cubes of the heights of the pyramids.*

For, if H, h be the heights, V, v the volumes,

vol. **OABCD** : $= \frac{1}{3}H \times$ area **ABCD**, and vol. **Oabcd** $= \frac{1}{3}h \times$ area **abcd**.

Now the areas **ABCD**, **abcd** are similar ;

$\therefore$ area **ABCD** : area **abcd** $=$ $AB^2 : ab^2 = OB^2 : Ob^2 = H^2 : h^2$,

$\therefore \quad V : v = H^3 : h^3$.

The same result may be proved similarly for a cone.

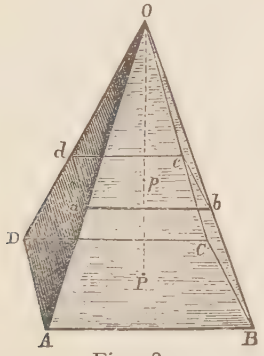


Fig. 183.

237. **To find the C.G. of a frustum of a pyramid or cone.** —Let **O** be the vertex, **P** the C.G. of the base of the pyramid. Then, if **OP** cuts the plane of section of the frustum (**abcd**) in **p**, the point **p** is the C.G. of the base of the upper pyramid cut off.

Hence the C.G.'s of the whole pyramid and that of the part cut off are on **OP**, and their distances from **O** are $\frac{3}{4}$**OP**

and $\frac{3}{4}$Op, respectively. Also the weights of the two pyramids are in the ratio of their volumes, and therefore as $OP^3 : Op^3$, since OP, Op are evidently either equal or at any rate proportional to their altitudes.*

Therefore the C.G. of the frustum is in OP, and its distance from O is given by

$$OG = \frac{OP^3 \times \frac{3}{4}OP - Op^3 \times \frac{3}{4}Op.}{OP^3 - Op^3}$$

Or, if a, b denote the lengths OP, Op, respectively,

$$OG = \frac{3}{4}\frac{a^4 - b^4}{a^3 - b^3} = \frac{3}{4}\frac{(a^2 + b^2)(a + b)}{a^2 + ab + b^2}.$$

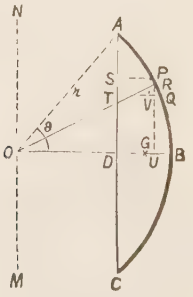


Fig. 184.

238. **To determine the centre of gravity of a circular arc.** —Let ABC be a uniform wire in the form of a circular arc (Fig. 184) ; let B be the middle point, and O the centre of the circle. Join OB, AC, and draw NOM perpendicular to OB, and, therefore, parallel to AC. Let G be the centre of gravity, which lies on OB (by symmetry). Let r, c, and

* In a *right cone* or *regular pyramid* OP is perpendicular to the base, and OP, Op are the altitudes of the whole and the upper part.

s be the lengths of the radius, the chord **AC**, and the arc **ABC**, respectively.

Now suppose the wire divided into a large number of parts, each of which is so small that it may be regarded as a straight line. Let **PQ** be one of these parts, and **R** its middle point. Draw **PS**, **QT** perpendicular to **AC**, and **PV**, **RU** perpendicular to **QT**, **OB**, respectively. Join **OR**.

Let the weight of unit length of the wire be w. Then the weight of the arc $\mathbf{PQ} = w \,.\, \mathbf{PQ}$; and this weight may be collected at **R**; also the distance of **R** from **MN** = **OU**.

Thus **OG** = distance of the centre of gravity from **MN**

$$= \frac{\Sigma\{(w \,.\, \mathbf{PQ}) \,.\, \mathbf{OU}\}}{\Sigma(w \,.\, \mathbf{PQ})} = \frac{\Sigma(\mathbf{PQ} \,.\, \mathbf{OU})}{\Sigma(\mathbf{PQ})} \quad \text{(A)}.$$

(Cf. § 174.)

But $\Sigma(\mathbf{PQ}) =$ arc **ABC** $= s$.

Again, since the sides of the triangle **PVQ** are respectively perpendicular to the sides of the triangle **OUR**, therefore these two triangles are similar;

thus $\mathbf{PQ} : \mathbf{PV} = \mathbf{OR} : \mathbf{OU}$;

whence $\mathbf{PQ} \,.\, \mathbf{OU} = \mathbf{PV} \,.\, \mathbf{OR}$.

Thus
$$\Sigma(\mathbf{PQ} \,.\, \mathbf{OU}) = \Sigma(\mathbf{PV} \,.\, \mathbf{OR}) = r\Sigma(\mathbf{PV}) = r\Sigma(\mathbf{ST}) = r \,.\, \mathbf{AC} = rc.$$

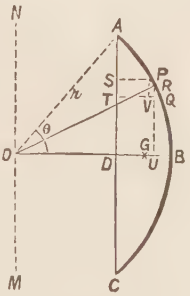


Fig. 184.

Thus, from (A), $\mathbf{OG} = rc/s$ .....(1).

Let 2θ be the circular measure of the angle subtended by the arc at **O** so that $\angle \mathbf{BOA} = \theta$. Then $c = 2\mathbf{AD} = 2r \sin\theta$, and arc $s = 2r\theta$.

Therefore $\mathbf{OG} = r \dfrac{\sin\theta}{\theta}$ .....(1a).

[When θ is very small, $\sin\theta/\theta = 1$ by Trigonometry, and, therefore, $\mathbf{OG} = r$, as it should be.]

Cor.—The C.G. of a semi-circular arc is found by putting $\theta = \frac{1}{2}\pi$, and, therefore, $\mathbf{OG} = 2r/\pi$.

239. To find the centre of gravity of the area of a sector of a circle.—Let **AOP** be the sector, and let its radius be r. Divide **AP** into a number of equal arcs **AB, BC, CD,** so small that each may be regarded as (approximately) straight. By joining the points of section to **O**, the sector

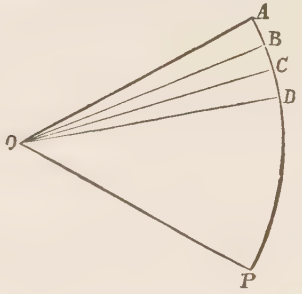


Fig. 185.

will be divided into narrow equal isosceles triangles **AOB, BOC,** . . . .

Now the C.G. of each of these triangles is at a distance from **O** of $\frac{2}{3}$ its median line (§ 168), that is, $\frac{2}{3}r$. Hence these C.G.'s all lie on a circular arc of radius $\frac{2}{3}r$. Replacing each triangle by a particle of the same mass at its C.G., the whole sector will be replaced by a number of equal particles evenly distributed along this circular arc, and when the bases **AB, BC,**... are made infinitely small, this distribution of particles becomes a *uniform* circular arc with centre **O** and radius $\frac{2}{3}r$, and of the same angle as the original sector. This arc, therefore, has the same C.G. as the original sector, and the distance of this C.G. from **O**

$$= \frac{2}{3} r \frac{\sin \theta}{\theta} \quad \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

2θ being the angle of the sector expressed in radians.

COR. 1.—**The C.G. of a semi-circular area** is found by putting $2\theta = \pi$ and $\therefore \sin \theta = 1$; hence its distance from the centre $= 4r/3\pi$.

COR. 2.—*The* C.G. *of a segment of a circle* **ABC** (Fig. 184) is the portion which is left when the triangle **OAC** is cut away from the sector **OABC**. Hence the C.G. can be found by the formula of § 235.

240. Before finding the centre of gravity of the surface of any zone of a sphere, it will be necessary to prove the following lemma :

LEMMA.—**The surface area of any zone of a sphere** *cut off between two parallel planes is equal to that which the planes intercept on a circumscribing cylinder to whose axis they are perpendicular.*

Fig. 186 represents the sectional view of a sphere, with a cylinder **CDD'C'** circumscribing it, whose axis coincides with the radius **OA** of the sphere. **H, K** are two points on **OA** which we shall firstly suppose to be very near to one another ; through **H, K** two planes are drawn both perpendicular to **OA.** We have to prove that the area of the zone of the sphere included between these two planes is equal to the area of the zone of the circumscribing cylinder included between them.

Firstly, to find the area of the zone of the circumscribing cylinder, we have

breadth of zone = **EF** = **HK** ;

distance round the zone

= circumf. of a circle of rad. **EH**

$= 2\pi$. **EH** $= 2\pi$. r ;

$\therefore$ area $= 2\pi$. $r \times$ **HK**.

Secondly, to find the area of the zone of the sphere, we have

breadth of zone = **PQ** ;

distance round the zone

= circumf. of a circle of rad. **RT**

$= 2\pi$. **RT** ;

$\therefore$ area $= 2\pi$. **RT** $\times$ **PQ**.

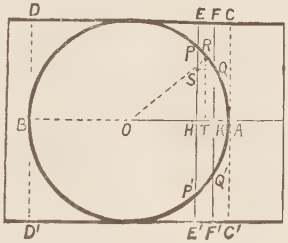


Fig. 186.

But, by similar $\triangle$s, **PSQ, OTR,**

PQ : **QS** = **OR** : **RT** ;

$\therefore$ **PQ** . **RT** = **QS** . **OR** = **HK** . r.

This proves that the two areas are equal.

When the bounding planes **EE** , $\mathbf{FF_1}$ are not very near together (Fig. 187), the zones may be divided into very narrow slices by drawing a number of parallel planes at small distances apart, thus dividing the axis **HK** into an infinitely large number of infinitely small portions. Each small zone of the sphere is equal to the corresponding zone of the cylinder, and therefore the same is true for the entire zones.

COR.—By dividing the whole area of the sphere into narrow zones by a series of planes perpendicular to **OA,** we can show that the area of the sphere is equal to the corresponding area of the circumscribing cylinder ; and, since the height of the corresponding portion of the cylinder is $2r$, this area becomes $2\pi r \times 2r = 4\pi r^2$.

241. To determine the centre of gravity of a zone of the surface of a sphere.

Suppose Fig. 186 to represent a thin spherical shell, and a circumscribing cylinder made of exactly the same material. Then, since the areas of the narrow zone of the sphere and the narrow zone of the cylinder are equal, their masses are also equal; also they have the same centre of gravity, viz., **T**.

It follows that in any problem dealing with the centre of gravity we can replace the mass of the narrow spherical zone by the mass of the corresponding cylindrical zone.

Now, in Fig. 187, we are required to find the centre of gravity of the spherical zone **PQQ'P** included between the planes through **P** and **Q** perpendicular to **OA**.

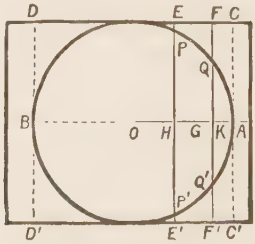


Fig. 187.

Divide this zone up into a large number of narrow zones by planes perpendicular to **OA**; we can then replace each narrow spherical zone by the corresponding zone on the cylinder without altering its mass or the position of its C.G. Thus we see that the centre of gravity of the spherical zone **PQQ'P'** is the same as that of the cylindrical zone **EFF'E'**; and the centre of gravity of the latter is obviously **G**, the middle point of **HK**.

Thus the centre of gravity of a spherical zone is the middle point of its axis..... (3).

COR.—The C.G. of a hemispherical bowl is at a distance of *half the radius* from the centre, as the same argument proves.

242. To find the C.G. of a sector of a solid sphere.

[DEFINITION.—A **sector** of a sphere is the portion of a solid sphere bounded by the surface of a (right circular) cone whose vertex is the centre of the sphere. (Fig. 188.)]

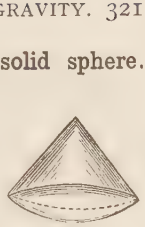


Fig. 188.

Draw a network of lines, dividing up the spherical surface of the sector into areas so small that each is approximately a plane area. These small areas will subtend at the centre of the sphere pyramids of which the sector is built up, and, if r is the radius of the sphere, the C.G. of each sector is at a distance $\frac{3}{4}r$ from the centre.

Hence each pyramid may be replaced by an equal mass at its C.G., and these masses will lie on a spherical cap of radius $\frac{3}{4}r$, bounded by the cone which bounds the sector. Moreover, each mass is proportional to the area of the base of the pyramid, and therefore also to the area on which it stands on the spherical cap of radius $\frac{3}{4}r$, proving that the distribution of matter over this cap is *uniform. Hence the* C.G. *of the sector is the* C.G. *of the corresponding uniform spherical cap on a concentric sphere of radius* $\frac{3}{4}r$ . . . . . (4).

It may be constructed as follows :

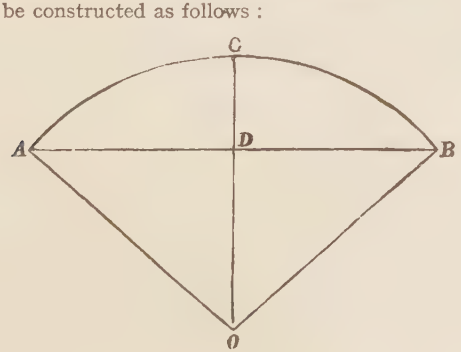


Fig. 189.

Let **AOBC** represent the sector seen in Fig. 188.

Then the C.G. of the surface of the cap **ACB** is at the middle point of **CD**, and its distance from **O** is $= \frac{1}{2}(OC + OD)$. (See § 240). Also

the C.G. of the sector, being the C.G. of a cap of $\frac{3}{4}$ the radius, is at a distance from **O** of $\frac{3}{4}$ this amount.

Hence, if **G** be the required C.G.,

$$\mathbf{OG} = \tfrac{3}{8}(\mathbf{OC} + \mathbf{OD});$$

or, if $\theta = \angle \mathbf{COA}$ = semi-vertical $\angle$ of bounding cone **AOB**,

$$\mathbf{OG} = \tfrac{3}{8}r(1 + \cos\theta).$$

COR. 1. **The C.G. of the volume of a hemisphere** is at a distance from the centre of $\frac{3}{8}$ *the radius.*

COR. 2. *The* C.G. *of the volume of the segment of the sphere* **ACBD** may be found by regarding it as the portion left when the cone **AOBD** is removed from the sector **AOBC**, the volumes and C.G.'s of the whole and the removed part being known.

243. To discuss the condition for stable equilibrium when a body on a rough curved base is standing on a rough curved surface, if there is contact at one point only.

[A special case of this problem would occur in the question, " Can a glass hemisphere stand on a fixed glass sphere ? "]

In the first place equilibrium is only possible if the C.G. of the body is vertically above the point of contact.

If this condition is satisfied, the equilibrium will be stable if the body, when tilted slightly from the equilibrium position, tends to return to that position. The question is closely allied to that which we have already discussed in § 161.

The problem is obviously very complicated, as it must involve the shape and relative position of the two surfaces in contact, as well as the position of the C.G. of the body. It can, however, be solved without very advanced mathematics if both surfaces are spherical, and if in the equilibrium position the common tangent plane at the point of contact is horizontal.

Let **HCK** (Fig. 190) be the spherical surface of the upper body in the equilibrium position, and **ACB** the rough spherical surface on which the body is standing : **G**, the centre of gravity of the body, **Q, P** the centres of the spheres. [In the diagram the contour of the body is not completed to avoid complicating the figure.] Let us suppose the body tilted by depressing **K**, till **H, C, K, Q, G** come into the positions **H', C , K', Q', G** respectively ; and let the new point of contact be **N**. Also let the radius $\mathbf{PC} = \mathbf{PN} = R$, the radius $\mathbf{QC} = \mathbf{QN} = r$, and the height of the centre of gravity above the point of contact $= \mathbf{GC} = \mathbf{GC} = h$.

The surfaces are rough, and we therefore suppose the body to roll without slipping. Hence arc **CN** = arc **CN**. But if θ and ϕ are the circular measures of the angles **CPN** and **CQN** respectively, the arc $\mathbf{CN} = R\theta$, and the arc $\mathbf{CN} = r\phi$. [*Tutorial Trigonometry*, § 16.

Thus $R\theta = r\phi$ or $\theta : \phi = r : R$ .....(i.)

Now, in the accompanying diagram the equilibrium is unstable,—for, in the displaced position of the body the line of action of the weight, DG′, falls to the right of N, and hence the weight of the body tends to move it farther from the equilibrium position,—but the equilibrium would be stable if the line of action DG′ fell to the left of N.

It follows that for stable equilibrium it is necessary that Q D should be greater than Q E, *i.e.* that the *horizontal projection of* Q G′ *should be greater than the horizontal projection of* Q N.

But the inclination of Q′N to the *vertical* is NQ Z or θ, and therefore its *horizontal* projection is $r \sin \theta$. Similarly the horizontal projection of Q G′ is Q′G′ sin G′Q Z or $(r - h) \sin (\theta + \phi)$. Thus for the condition of stable equilibrium we obtain

$(r - h) \sin (\theta + \phi) > r \sin \theta$ ....(ii.)

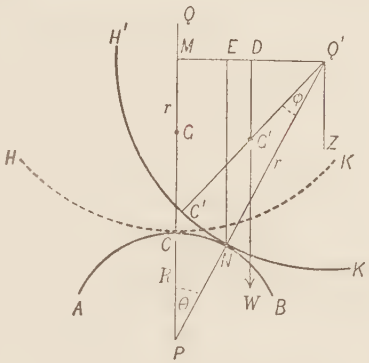


Fig. 190.

But θ and ϕ are very small, for we consider only a *slight* displacement (§ 159); also we may replace the sine of a small angle by its circular measure (*Tutorial Trigonometry*, § 247). Thus from (ii) we obtain the condition

$$(r - h)(\theta + \phi) > r\theta \quad \text{.....(iii).}$$

i.e. $$\frac{r - h}{r} > \frac{\theta}{\theta + \phi};$$

whence, using (i.), $$\frac{r - h}{r} > \frac{r}{r + R}.$$

Simplifying this we have $rR > hr + hR$,

or $$\frac{1}{h} > \frac{1}{R} + \frac{1}{r} \quad \text{.....(5)}$$

which is the condition required.

NOTE 1.—The last few steps may be replaced by any method of eliminating ϕ between (i.) and (iii.).

NOTE 2.—The result does not contain θ or ϕ, hence the question of stability is independent of the exact magnitude of the displacement, so long as it is slight.

NOTE 3.—The result applies equally to two cylindrical surfaces.

NOTE 4.—The result applies also to any surfaces which may be regarded as spherical in the immediate neighbourhood of the point of contact. This includes all surfaces of revolution having **QCP** for axis.

NOTE 5.—If **HCK** is a plane r is infinite, and the condition becomes $\frac{1}{h} > \frac{1}{R}$, *i.e.*, $h < R$. Similarly if **ACB** is a plane we have $h < r$ (cf. § 161).

NOTE 6.—If one of the surfaces is *concave*, we obtain the condition of stability by changing the sign of the corresponding radius in (5).

244. *Example.*—**AD** is a fixed smooth horizontal rod; a string **ABC** is fastened at **A**, and carries a body **B** of weight W at its middle point; the extremity **C** is attached to a ring which slides on the rod and which is acted on by a constant force P in the direction **AD**. Determine the position of equilibrium and show that it is stable.

Let θ be the inclination of each half of the string to the horizon in the equilibrium position.

B is in equilibrium under the two tensions T, T and the weight W.

Resolving vertically we obtain

$$W = 2T \sin\theta \quad \ldots\ldots \text{(i.)}$$

C is in equilibrium under the forces T, P, R.

Resolving horizontally we obtain

$$P = T\cos\theta \quad \ldots\ldots \text{(ii.)}$$

Eliminating T from (i.) and (ii.) we have

$$\tan\theta = \frac{W}{2P} \quad \ldots\ldots \text{(iii.)}$$

which determines θ, and therefore the equilibrium position.

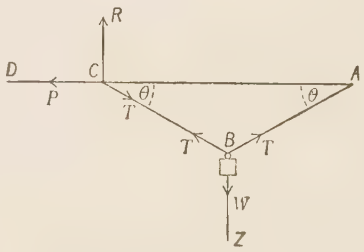


Fig. 191.

To discuss the stability of the equilibrium :—

From (iii) we have $P = \frac{1}{2}W\cot\theta$ ..... (iv):

Let us suppose that by a slight displacement of the ring **C** the angle θ is changed to ϕ; and let us determine what force Q would be required instead of P to maintain equilibrium in the new position.

Obviously we may infer from (iv) that

$$Q = \tfrac{1}{2}W \cot \phi \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (v.)$$

Now if **C** were displaced to the left of the equilibrium position $\phi < \theta$, therefore $\cot \phi > \cot \theta$, and thus from (iv) and (v) we have $Q > P$. Hence P is too small a force to produce equilibrium in the new position, and therefore the ring **C** will slip back to the right, i.e., *towards the equilibrium position.*

Similarly if **C** were displaced to the right of the equilibrium position, we should have $\phi > \theta$, *i.e.* $\cot \phi < \cot \theta$, *i.e.* $Q < P$. Thus P would be too great for equilibrium in this position, and **C** would be pulled to the left, i.e., *again toward the equilibrium position*. Hence the equilibrium is stable.

Summary of Results.

Distance of the c.g. of a *circular wire* from the centre

$$= rc/s = r \sin \theta/\theta \quad \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

For a *circular sector* the distance of the c.g. from the centre is $\frac{2}{3}$ that of the arc on which the sector stands. . . . (2).

Centre of gravity of a *spherical zone* is at the middle point of its axis. . . . . (3).

For a *spherical sector* the distance of the c.g. from the centre $= \frac{3}{4}$ that of the spherical cap on which the sector stands. . . . . (4).

Condition for the equilibrium of a body with a curved base on a curved surface is

$$\frac{1}{h} > \frac{1}{R} + \frac{1}{r} \quad \ldots\ldots\ldots\ldots\ldots\ldots (5).$$

EXAMPLES XVIII.

(*Selected Problems* :—1, 3, 4, 7, 8, 9, 12, 13, 14, 16, 18, 20, 21, 23, 24.)

1. Explain how to find the c.g. of a segment of a circle greater than a semi-circle.

2. Determine the centre of gravity of a homogeneous hemisphere of radius r, from which a concentric hemispherical portion of radius $\frac{1}{2}r$ has been scooped out.

3. Prove that if a segment be cut off from a solid sphere of radius a by a plane distant b from the centre the c.g. of the segment is at a distance from the centre of $\dfrac{3(a+b)^2}{4(2a+b)}$.

4. If a cone be placed with its base in contact with a smooth inclined plane, and be prevented from slipping by a horizontal force applied at the apex, give a diagram showing the arrangement of the forces acting on it.

Also, if the inclination of the plane be 45°, and the radius of the base be equal to ·375 of the altitude, prove that the cone will be on the point of toppling over.

5. A truncated cone is bounded by plane faces, perpendicular to the axis, the diameters 4 ft. and 5 ft., respectively, and rests with its smaller face upon a horizontal plane. A uniform plank of length 6 ft. is laid upon the upper surface, its middle point coinciding with the centre of the latter, and is just tilted by a vertical force at one of its extremities. Prove that the truncated cone will not be moved provided its weight be not less than $\frac{3}{2}$ of the weight of the plank.

6. Find the centre of gravity of a quadrilateral lamina, two of whose sides are parallel and one of them double of the other, calculating its distance from the point where the other pair of sides meet when produced.

7. The uniform quadrilateral **ABCD**, right-angled at **A** and obtuse-angled at **B**, is bisected by the diagonal **AC**. If it rest with the side **AB** on a horizontal plane, prove that **BM** must be less than $\frac{3}{2}$**AB**, where **M** is the foot of the perpendicular from **C** on **AB** produced through **B**.

8. Prove that a thin hemispherical bowl can rest in equilibrium with its curved surface in contact with a rough inclined plane, if the inclination does not exceed 30°.

9. Three uniform wires of different densities form a triangle **ABC**. If the centre of gravity is at the centre of the inscribed circle show that the masses of the wires are in the ratios

$$\cot \frac{A}{2} : \cot \frac{B}{2} : \cot \frac{C}{2}.$$

10. A circular lamina is hung up from three points **A, B, C** on its rim by equal strings attached to a fixed point. Show that the tensions of the strings are as

$$\sin 2A : \sin 2B : \sin 2C.$$

11. The angle of a sector is 60°, and the part of it within the inscribed circle is removed. Find the centre of gravity of the remainder.

12. Show that the centroid of a quadrilateral lamina coincides with the centroid of four equal particles at the angular points and one particle of equal but negative mass at the intersection of the diagonals.

13. If forces acting from a point **O** are represented completely by $m_1\mathbf{OA}_1$, $m_2\mathbf{OA}_2$, etc.... show that their resultant is represented by $(m_1 + m_2 \ldots.)$ **OG** where **G** is the centre of gravity of masses $m_1, m_2, \ldots.$ placed at $\mathbf{A}_1, \mathbf{A}_2, \ldots.$

D is a point in the plane of the triangle **ABC**, **I** the centre of its inscribed circle. Show that the resultant of forces acting through **D** and represented by $a.\mathbf{AD}$, $b.\mathbf{BD}$, $c.\mathbf{CD}$ is $(a + b + c)$ **ID**.

14. A portion of a uniform thin spherical shell is bounded by a circular rim. When the shell is freely suspended from a point in the rim, the radius through the lowest point in the rim is horizontal. Show that the plane bounding the shell is at a distance from the centre of the sphere equal to two-thirds of the radius.

15. A crescent is bounded by the arcs of equal circles (of radius r). If $2r\cos\theta$ is the distance between the centres, show that the distance of the centre of gravity of the crescent from the centre of the inner curve is

$$r\cos\theta\left\{1 + \frac{\pi}{\pi - 2\theta + \sin 2\theta}\right\}.$$

16. A hemisphere, of radius a and weight W, is placed with its curved surface on a table, and a string of length l $(l < a)$ is attached to a point on its rim and to a point on the table. Prove that, if the table be smooth, the tension of the string is

$$3W(a - l)/8\sqrt{(2al - l^2)}.$$

17. A half-hoop in the exact form of a semicircle rests on a rough ground against a smooth wall, its plane being perpendicular to the ground and to the wall. Find its position when it is on the point of slipping.

18. A semicircular cylinder rests with its axis horizontal and with its curved surface on a rough inclined plane of inclination α. Find the inclination of its plane surface to the horizontal in the position of equilibrium.

19. What is the condition that a solid right circular cone can stand symmetrically in stable equilibrium on the top of a fixed rough sphere of radius r?

20. Find the radius of the largest solid hemisphere which can remain in stable equilibrium on the top of a rough fixed sphere of radius a, (i) if it rests on the middle point of its curved surface, (ii) if it rests on the middle point of its plane surface.

21. A light sphere of radius r carries a weight W at its highest point; find the greatest possible radius of a rough hemispherical bowl at the bottom of which this sphere can rest in stable equilibrium.

22. Prove that a plane lamina can remain in stable equilibrium if it rests with its centre of gravity in contact with *any point* on the upper half of the surface of a rough fixed sphere.

23. If in the figure of § 13 the pulleys **H** and **K** are on the same level, and the weights P and Q are equal, prove that the equilibrium is stable. Consider only vertical displacements of R.

24. **A**, **B** are two points on the same level at a distance a. A uniform rod **AC** of length a and weight $2\sqrt{3}$ lb. is hinged at **A** and is supported by a string fastened at **C**, passing over a pulley at **B**, and carrying a weight of one pound. Determine the equilibrium position, and show that the equilibrium is stable. [Displace it through a small angle α; then use the equations $\cos\alpha = 1$, $\sin\alpha = \alpha$.]

25. A uniform rod is suspended from a smooth peg by a string passing over the peg and attached to the ends of the rod. The rod can rest either horizontally or vertically. Which of these positions is stable. [Consider the tensions if the rod is suspended by unequal strings *tied* to the peg and to its extremities.]

EXAMINATION PAPER VIII.

1. State the laws of friction, and explain what is meant by *limiting friction, coefficient of friction, angle of friction.*

2. A rigid body of any form being supposed to rest on a rough horizontal plane; given the coefficient of friction between it and the plane, determine, in direction and magnitude, the least force which, acting through its centre of gravity, will just move it.

3. A body of weight W is held in equilibrium on a plane of inclination $37\frac{1}{2}°$ by a force applied horizontally. If the angle of friction be $7\frac{1}{2}°$, construct a triangle of forces, from which (or otherwise) calculate the greatest and least values of the horizontal force consistent with equilibrium.

4. A uniform rod of given length is to be supported in a given inclined position, with its upper end resting against a rough vertical wall, by means of a string attached to the lower end of the rod and to a point of the wall. Find, by a geometrical construction, the highest and lowest points of the wall to which the string can be attached.

5. Prove, by any method, the condition for equilibrium in the common screw press, friction being taken into account.

6. Find the centre of mass of the area of a quadrilateral the coordinates of the angular points of which are given.

7. Find the centre of mass of a uniform thin bowl of hemispherical form.

8. From the position of the C.G. of a hemispherical shell deduce the position of the C.G. of a solid hemisphere.

9. Prove that, if a homogeneous hemisphere is joined to a cylinder on the same base and of the same material, the equilibrium will be stable when the hemisphere is resting on a horizontal plane if the radius of the base is greater than $\sqrt{2}$ times the height of the cylinder.

10. In the so-called "first" and "third" systems of pulleys (the two systems in which each pulley is worked by a separate string), prove that the mechanical advantages are reduced to $(1 + m)^n$ and $(1 + m)^n - 1$, when it is found that the friction of the rope and pulleys causes the tension to be reduced to m times its value in passing round a pulley.

MISCELLANEOUS EXERCISES.

(*Selected Problems* :—1, 2, 6, 7, 9, 10, 12, 14, 18, 20, 22, 23, 25, 26, 29, 31, 33, 35, 36, 42, 43, 47, 48, 50, 51, 54, 56, 59, 60, 61, 63, 65, 66, 68, 71, 73, 75, 76, 79, 82, 85, 91, 95, 97, 111, 112, 113, 114.)

1. The centre of the circumscribed circle of a triangle **ABC** is **O**, and the intersection of the perpendiculars from the angular points on the opposite sides is **P** ; prove that the resultant of forces **OA, OB, OC** will act along **OP**, and be equal to **OP**.

2. Four pegs are fixed in a wall at the four highest vertices of a regular hexagon (the two lowest being in a horizontal straight line), and over these is thrown a loop suspending a weight. The loop has such a length that the angles formed by it at the lowest pegs are right. Determine the tension in the string and the pressures on the pegs.

3. A bar of uniform thickness and density, 12 ft. long and 1 cwt. in mass, is supported at its extremities in a horizontal position ; if a body of 2 cwt. be suspended from a point 2 ft. distant from one end and a body of 4 cwt. at 4 ft. from the other end, find the pressure on the points of support.

4. A rigid body of any form, supported at three points of its surface by three vertical props, is in equilibrium under the action of gravity. All particulars being supposed given, required the pressures P, Q, R on the props.

5. A uniform straight bar, of length l and weight W, suspended from a fixed point by two cords, of lengths a and b, attached to its extremities, is in equilibrium under the action of gravity. Required the tensions P and Q on the cords.

6. An equilateral triangle, whose sides are light rods hinged freely together, stands on a horizontal plane. A weight W is suspended from the vertex ; determine the tension of the base.

7. Determine the tension of the base in the last question if there are three weights W suspended, one from the vertex, and one from the middle point of each side.

8. Four equal heavy rods are hinged at their extremities to form a rhombus, two opposite corners of which are joined by a string; if the system be suspended from one of the angular points to which the string is attached, find the tension of the string.

9. Five equal light rods, freely hinged together, form a framework in the shape of a rhombus **ABCD**, with a diagonal **BD**. This carries a weight, W, at **D**, and is supported by two vertical strings at **A** and **C** in such a way that **AB** and **CD** are horizontal, **CD** being on the lower level. Determine the tension along **BD**.

10. **ABCD** is a light string tied to two fixed points at **A** and **D**, and carrying weights at **B** and **C**. Determine the ratio of the weights at **B** and **C** if **AB** is inclined at 30° to the vertical, **BC** at 30° to the horizontal, and **CD** is horizontal.

11. A series of equal weights are knotted at different points of a string the two extremities of which are tied to two fixed points. Prove that the tangents of the inclinations to the horizontal of the successive portions of the string are in A.P.

12. A regular hexagon **ABCDEF**, made of light rods freely hinged together, carries equal heavy weights, W, at the angular points; if the side **AB** be held horizontally, and the figure kept in shape by a light rod **CF**, find the thrust along **CF**.

13. If the regular hexagon in the last question were kept in shape by two strings **AD**, **BE**, instead of the rod **CF**, determine the tension in each of the strings.

14. **BC**, **CA**, **AB** are three weightless rods formed into a triangular frame; their lengths are respectively 10, 8, 6; the frame is suspended from the angular point **A**; a weight of 100 lb. is hung from the middle point of **BC**. Find the stresses in **BC**.

15. What would be the answer to the last question if, instead of the 100 lb. weight, two weights of 50 lb. each were hung from **B** and **C**?

16. Draw a rectangle **ABCD** and its diagonals **AC**, **BD** intersecting at **E**, the lengths **AB** and **AC** being 6 and 10 ft. respectively; let its plane be vertical, and **AB** horizontal.

AC and **BD** represent two weightless rods turning freely about a pin at **E**, with their lower ends **A**, **B** connected by a

thread, and standing on a horizontal plane. If a weight W is hung at **C**, find the pressures on the ground, the tension of the thread, and the stress in **BD**.

17. Two weightless levers **AOB**, **COD**, whose lengths are 8 and 9 in. respectively, are joined together at **O**, 4 in. from **B** and **D**. If **A**, **C** be connected by a string 3 in. long, and **B**, **D** be pulled apart by forces **P**, **P** in the straight line **BD**, find the tension of the string.

18. The string **ABCD**, attached to two fixed points **A** and **D**, has two equal weights knotted to it at **B** and **C**, and rests with the portions **AB** and **CD** inclined 30° and 60° respectively to the vertical. Prove that the tension in the portion **BC** is equal to either weight, and that **BC** is inclined 60° to the vertical.

19. **A**, **B**, **C** are three points in a vertical plane, **A** and **C** lying on opposite sides of the vertical line through the highest point **B**. A string **ABCDE**, having one end fixed at **A**, passes in succession through a light smooth ring **D**, round pegs at **B** and **C**, again through the ring, and is attached to a weight at its free extremity **E**. Prove that in the position of equilibrium the ring and the weight hang vertically below **B**.

20. Four equal weightless rods are hinged together to form the rhombus **ABCD**, and the hinges at **A** and **C** are connected by a string. If the rhombus be suspended from **A**, and equal weights of 1 cwt. each be suspended from **B** and **D**, find graphically the tension of the string.

21. Five equal weightless rods are hinged together so as to form the rhombus **ABCD**, and the diagonal **BD**. If the rhombus be suspended from **A**, and a weight of 1 cwt. be suspended from **C**, find the thrust in **BD**.

22. Three equal smooth spheres on a smooth horizontal plane are in contact with each other, and are kept together by an endless string in the plane of their centres, just fitting round them. If a fourth equal sphere is placed upon them, prove that the tension of the string is to the weight of either sphere as $1 : 3\sqrt{6}$.

23. Two equal weightless rigid bars **AB** and **AC**, hinged together at **A**, and connected by a weightless string **BC**, are placed in a vertical plane, with **B** and **C** upon a smooth horizontal plane. If a given

weight be suspended from a point **O** in **AC**, find, graphically, the action at **A**, and the tension of the string. If **O** be moved along **AC**, prove that this tension varies as **OC**.

24. A uniform triangular lamina, **ABC**, right-angled at **B**, is standing on its side **BC**, on a horizontal plane. If the plane be gradually tilted, so that **C** is higher than **B**, prove that, all sliding being prevented, the greatest possible angle of inclination, consistent with there being no toppling over, is equal to **A**.

25. A rectangular block **ABCD**, whose height is double its base, stands with its base **AD** on a rough floor ($\mu = \frac{1}{6}$). If it is pulled by a horizontal force at **C** till motion ensues, determine whether it will slide or topple.

26. A lamina hangs freely in a vertical plane with a point **A** fixed, and in this position a horizontal line **BC** is drawn upon it; it next hangs with **B** fixed, and a horizontal line is drawn through **A**, cutting **BC** in **C**. Show that, if **C** be fixed, the lamina will rest with the line **AB** horizontal.

27. Any closed polygon is drawn with $2n$ sides; **E, F, G**, etc., are the points of bisection of n alternate sides; **e, f, g**, etc., are the points of bisection of the remaining n sides. Prove that the C.G. of any n equal weights at **E, F, G**, etc., coincides with that of any n equal weights at **e, f, g**, etc.

28. If four forces acting along the sides of the plane quadrilateral **ABCD** be in equilibrium, and one of them be completely represented by **AE**, where **E** is a given point in **AB** or **AB** produced, and if the quadrilateral **AEFG** be constructed, having its sides parallel and proportional to the four forces, prove that

$$\frac{AE}{AB} \cdot \frac{FG}{CD} = \frac{AG}{AD} \cdot \frac{EF}{BC}.$$

29. If **E** and **F** be the middle points of the diagonals **AC** and **BD** of the quadrilateral **ABCD**, and if **EF** be bisected in **G**, prove that the four forces **AG, BG, CG**, and **DG** will be in equilibrium.

30. Three forces, **PA, PB, PC**, diverge from the point **P**; and three others, **AQ, BQ, CQ**, converge to the point **Q**. Prove that the resultant of the six is represented in magnitude and direction by $3\mathbf{PQ}$, and passes through the C.G. of the triangle **ABC**.

31. It is required to decompose a force, whose magnitude and line of action are given, into two equal forces passing through two given points. Give a geometrical construction for solving the problem (i) when the two points are on the same side of the line, (ii) on opposite sides.

32. **ABCD** is a uniform quadrilateral lamina; **E, F, G, H** are the C.G.'s of the triangles **BCD, CDA, DAB, ABC**. Prove that the quadrilateral **EFGH** is similar to the quadrilateral **ABCD** turned through two right angles.

33. The uniform quadrilateral **ABCD**, right-angled at **A**, and obtuse-angled at **B**, is bisected by the diagonal **AC**. If it rest with the side **AB** on a horizontal plane, prove that **BM** must be less than $\frac{3}{2}$ **AB**, where **M** is the foot of the perpendicular from **C** on **AB** produced through **B**.

34. Forces P, Q, R act along the sides **BC, CA, AB** of a triangle. Prove that their resultant will act along the line joining the centre of the circumscribing circle to the orthocentre if

$$P : Q : R = \frac{\cos \mathbf{B}}{\cos \mathbf{C}} - \frac{\cos \mathbf{C}}{\cos \mathbf{B}} : \frac{\cos \mathbf{C}}{\cos \mathbf{A}} - \frac{\cos \mathbf{A}}{\cos \mathbf{C}} : \frac{\cos \mathbf{A}}{\cos \mathbf{B}} - \frac{\cos \mathbf{B}}{\cos \mathbf{A}}.$$

35. A uniform ring rests with its plane horizontal over the top of a smooth fixed sphere. What is the nature of the equilibrium?

36. A weightless triangular lamina **ABC** is supported at the three angular points in a horizontal position. A weight W is placed at its orthocentre. Determine the pressures on the support at **A**.

37. Find the pressure at **A** in the last question, if the weight be placed at the centre of the circumscribing circle.

38. Two equal smooth spheres, each of weight W, are suspended by parallel weightless strings of equal lengths, the line joining their centres being horizontal, when an isosceles triangle of weight W is thrust between them, with its vertical angle 2α downwards. If β be the inclination of either string to the vertical when there is equilibrium, prove that $\cot \beta = 3 \tan \alpha$.

39. If **OA, OB, OC** are three diverging bars of the same material and of the same section, prove that, if their C.G. is at **O**, the sines of the angles they make are as the squares of their lengths.

40. Three forces P, Q, R acting at a point are in equilibrium. Upon their respective lines of action three points **A, B, C** are taken, and P is replaced by its components in **AB** and **AC**; Q by its components in **BA** and **BC**; R by its components in **CA** and **CB**. Prove that the two forces thus acting in each of the lines **AB, BC, CA** are equal and opposite.

41. Three smooth pegs, **A, B,** and **C,** are fixed in the same vertical plane, **B** being 3 ft. vertically above **A,** and **C** 4 ft. horizontally distant from **A.** A weightless string 13 ft. long passes round **A, B,** and **C,** and is attached at its extremities to a given weight W. Find the tension of the string and the pressure on the peg **B** in terms of W.

42. Two particles whose weights are equal are connected by a light string, and rest on a rough vertical circle; the coefficient of friction being $\tan \epsilon$. Show that when there is no friction between the string and the circle, and both weights are on the point of motion, the radius of the circle through the middle point of the string will make an angle ϵ with the vertical.

43. A wheel, weight W and radius r, rests between two planes each inclined to the vertical at angle α. The plane of the wheel is perpendicular to the line of intersection of the two planes, which is itself horizontal. If μ be the coefficient of friction, find the least couple necessary to turn the wheel.

44. A uniform beam of weight W, laid on a horizontal plane, can be just moved by pushing it with a horizontal force $\frac{1}{3}W\sqrt{3}$. Prove that the least force which can move it is equal to $W/2$, and find the direction of this force.

If the beam be pulled slowly by a rope attached at one end **A,** prove that **A** will not rise from the ground unless the inclination of the rope to the horizon be 60° at least.

45. A cube is placed with one edge on a rough horizontal plane (coefficient of friction $= \mu$) and a parallel edge on a smooth plane, inclined at an angle of 45° to the horizon; if θ is the inclination of the base of the cube when in the position in which it will just not slide into a lower position, show that $(1 + 3\mu) \tan \quad = 1 - \mu$.

46. A body of given weight W is urged against a rough inclined plane by a force P acting in a direction at right angles to the plane.

Under what circumstances will P keep the body from sliding down the plane? The inclination of the plane to the horizon being 30°, and the weight of the body 20 lb., what must be the coefficient of friction if the force, acting as above, which just keeps the body from sliding is 10 lb.?

47. It is found by trial that, when P is on the point of lifting Q by means of a single fixed pulley, $P = (1 + m)Q$, where m is a constant depending on the friction of the various parts of the machine. If three such pulleys are combined into a block and tackle, find the power requisite to raise a given weight by means of it. If $m = 0{\cdot}2$, find the number of ft.-lb. of work done against friction when a weight of 1000 lb. is raised 20 ft. by means of a block and tackle of three such pulleys.

48. Two weights, P and Q, hang by a flexible thread over a fixed pulley, capable of turning on a rough axle; find the relation between them when P is on the point of moving downwards. If the radii of the pulley and axle are 10 in. and 1 in., respectively, and if a weight of 101 lb. is on the point of motion when counterbalanced by a weight of 99 lb., find the coefficient of friction between the axle and its bearing.

49. Three equal rough spheres of the same material rest in contact on a horizontal plane, and support another similar sphere on the top of them. Calculate the frictions at the points of contact of the upper sphere, and show that, if the equilibrium is disturbed, it will be by the three lower spheres rolling along the plane.

50. Two equal and similar rough cylinders, whose axes are parallel, and which rest in contact on a rough horizontal plane, support another equal cylinder (whose axis is parallel to theirs) on the top of them. Calculate the frictions at the points of contact of the upper cylinder; and show that, if the equilibrium is disturbed, it will be through the two lower cylinders rolling along the plane.

51. If **ABC**, **A′B′C′** be any two triangles, and **G**, **G′** their centres of gravity, show that forces at a point parallel and proportional to **AA′**, **BB′**, **CC′**, 3 **G′G** will be in equilibrium.

52. Particles with masses proportional to a sec A, b sec B, c sec C, are placed at the points **A**, **B**, **C** of $\triangle$ **ABC**; find their C.G.

53. A triangular lamina, of weight W, is supported by three vertical strings attached to its angular points, so that the plane of the triangle is horizontal. A particle of weight equal to W is placed on the triangle at its orthocentre. Prove that the tensions of the strings are given by

$$\frac{T_1}{1 + 3\cot B \cot C} = \frac{T_2}{1 + 3\cot C \cot A} = \frac{T_3}{1 + 3\cot A \cot B} = \frac{W}{3}.$$

54. Prove that the centre of gravity of a quadrilateral cannot coincide with the centre of gravity of four equal particles placed at its angular points, unless the quadrilateral is a parallelogram.

55. **AB, AC** are two uniform rods of lengths $2a$, $2b$; prove that, θ being the angle **BAC**, the distance from **A** of the centre of gravity of the two rods is

$$\sqrt{a^4 + 2a^2 b^2 \cos\theta + b^4}/(a + b).$$

56. A straight line is drawn joining the middle points of two sides of a plane triangular plate of uniform thickness. If the densities of the smaller and greater portions into which the plate is thus divided be respectively ρ and ρ', find its centre of gravity.

57. A frustum of a cone whose semi-vertical angle is α, is made by cutting off $1/n$th of the axis. Prove that the frustum will rest with a slant side on a horizontal plane if

$$\tan^2\alpha < \tfrac{1}{4}.\ \frac{3n^4 - 4n^3 + 1}{n^3 - 1}.$$

58. A body is bounded by two spheres (radii a, b) which touch internally. Find the centre of gravity, and show that in the limiting case, when $a = b$, it divides the common diameter in the ratio 2 : 1.

59. If there are masses m_1 and m_2 at points $\mathbf{P}_1$, $\mathbf{P}_2$ and if **G** is their centre of gravity and **O** any other point, prove that

$m_1 . \mathbf{OP}_1^2 + m_2 . \mathbf{OP}_2^2 = m_1 . \mathbf{P}_1\mathbf{G}^2 + m_2 . \mathbf{P}_2\mathbf{G}^2 + (m_1 + m_2)\ \mathbf{OG}^2$.

Deduce the corresponding theorem for three or more masses.

60. Forces P, Q, R, S act along the sides **AB, BC, CD, DA** of a quadrilateral which can be inscribed in a circle. Show that if their resultant is parallel to **AC**

$$P . \mathbf{BC} - Q . \mathbf{AB} = R . \mathbf{AD} - S . \mathbf{CD}.$$

61. Show that it is easier to pull a cart over an obstacle in the road if the wheel is large than if it is small.

62. A balance has scale-pans of different weight and arms of different length, but when the pans are empty the beam is horizontal and its centre of gravity is in the same vertical as the fulcrum. When weights P, Q are placed in the scale pans the inclination of the beam to the horizon is θ, and when the weights are interchanged it is ϕ (on the same side). Prove that the ratio of the arms is

$$P \tan\theta - Q \tan\phi : Q \tan\theta - P \tan\phi.$$

63. The centre of gravity of the beam of a balance of which the weight is W is at a distance c below its axis of suspension, and the axes of suspension of the scale-pans are at a distance d below the same axis, when the beam is in equilibrium. Prove that when the scale pans are removed a weight w, hung on to one of these axes, will make the beam incline to an angle whose cotangent is $(Wc + wd)/wa$, where $2a$ is the distance between the two axes.

64. In Fig. 148, § 184, if $OG = b$, $OH = c$, prove that the inclination of the beam to the horizontal is given by

$$\cot\theta = (2cQ + bW + cw)/aw.$$

65. A uniform board, in the shape of an isosceles triangle and of weight W, is suspended freely at its vertex; a small weight w is attached to one extremity of the base. Given the base $= 2a$, and the altitude $= h$, prove that the end to which the weight is attached is depressed a distance

$$3wa^2/[h(3w + 2W)] \text{ approximately.}$$

66. Two equal smooth cylinders of radius c rest symmetrically within a cylinder of radius a. Find the stress between the former cylinders at their point of contact.

67. A thin homogeneous smooth bottomless circular cylindrical shell of radius $a \cos\alpha$ rests on a smooth horizontal plane. A uniform rod of length $2a$ is placed in the cylinder with one end resting against the vertical side of the cylinder and the other touching the cylinder and the plane. Find all the reactions on the rod, and show that equilibrium will be impossible unless the weight of the cylinder exceeds that of the rod.

68. Two smooth spheres, each of radius r and weight W, are placed inside a hollow cylinder of radius a, open at both ends, which rests on a horizontal plane with its axis vertical. Prove that, in order that the cylinder may not upset, its weight must at least be

$$2W\left(1 - \frac{r}{a}\right)$$

given that $r > \frac{1}{2}a$.

69. Two thin boards in the shape of equilateral triangles, of side a and weight W, are leaning against one another at their vertices, with their bases on a smooth horizontal table, and the extremities of their bases connected by strings of length a. Find the tension of each string.

70. One corner of a vertical square lamina rests on a rough horizontal plane, and a string is attached to an adjacent corner and pulled horizontally. Show that, if the coefficient of friction is greater than $\frac{1}{2}$, any position in which the string is stretched is one of equilibrium.

71. A ladder 13 ft. long rests with one extremity on a rough pavement and the other against a smooth vertical wall at a height of 12 ft. from the ground. A man ascends it. Show by a diagram, or otherwise, that the frictional resistance of the ground must increase as he ascends, and that if the coefficient of friction be less than five-twelfths, the ladder, whose weight may be neglected, will slip before he reaches the top.

72. A uniform ladder of weight W, with one end on the ground, is lowered from a vertical position by a rope tied to its top, which passes under a small fixed pulley on the ground at a distance from the foot of the ladder equal to its length. The rope cannot stand a greater strain than $\frac{1}{2}W$. Prove that, if the coefficient of friction between the ladder and the ground be greater than $\frac{1}{5}\sqrt{3}$, the ladder can be lowered without slipping till the rope breaks, which will happen when the ladder is inclined at 60° to the ground.

73. Show that when the wheels of a railway carriage are on the point of being skidded by the brakes, a couple of moment $\mu W r$ is called into play, tending to tilt the body of the carriage on its springs in a lengthwise direction, μ denoting the coefficient of friction, W the total weight of the carriage, and r the radius of the wheels.

74. Two rough spheres of equal radii but unequal weights W_1, W_2, rest in a spherical bowl; their centres of gravity coincide with their centres, and the line joining these is horizontal and subtends an angle 2α at the centre of the bowl. Prove that the coefficient of friction between them is not less than

$$\frac{W_1 \sim W_2}{W_1 + W_2} \tan\left(\frac{\pi}{4} - \frac{\alpha}{2}\right).$$

75. A solid homogeneous hemisphere, of radius a and weight W, rests with its curved surface on a fixed horizontal plane, and a particle of weight W' is placed on it at a distance x from the centre. Prove that, in the position of equilibrium, the friction exerted between the particle and the hemisphere is equal to

$$8xW'^2(9a^2W^2 + 64\,x^2W'^2)^{-\frac{1}{2}}.$$

76. Two uniform rods **AB**, **BC** of equal length are connected by a smooth hinge at **B**, while **AB** can turn freely round a fixed point at **A**. The two are supported in a horizontal line by a prop under **BC**. Given the weights of the rods, find the magnitude and direction of the action at **B**, and the pressure on the prop. Is the position of the prop determinate?

77. **ABC** is a triangle, right-angled at **C**, and **P** is any point on its hypotenuse. Forces proportional to **PB** cos **PCB**, **PA** cos **PCA**, act along the sides **CA**, **CB**. Show that their resultant acts in a direction perpendicular to **AB** and find its magnitude.

78. Forces ka, kb, kc act along the sides a, b, c of a triangle, and a fourth force P is given in magnitude. Prove that, if the line of action of P touches a given circle of radius r, the line of action of the resultant will touch a fixed circle, and find its radius.

79. Two forces represented in magnitude and direction by **OA**, **OB** have a resultant represented in magnitude and direction by **OC**; the circle on **OC** as diameter cuts **OA**, **OB** in **A'** and **B'** respectively. Prove that $\mathbf{OC}^2 = \mathbf{OA} \cdot \mathbf{OA'} + \mathbf{OB} \cdot \mathbf{OB}$.

80. Forces P, Q, R act respectively along the lines **BC**, **CA**, **AB**; prove that, if S is the magnitude of their resultant,

$$S^2 = P^2 + Q^2 + R^2 - 2QR\cos A - 2RP\cdot\cos B - 2PQ\cdot\cos C.$$

81. Three equal particles, attracting one another directly as the distance, are capable of motion each along one side of a fixed triangle. When in equilibrium show that they are situated at the feet of the perpendiculars drawn from their centre of gravity to the sides.

82. Prove that an engine-driver will lose control of a train on an incline steeper than 1 in $n \cot \phi$, if the wheels which are braked carry only $1/n$ of the gross weight of the train, the angle ϕ denoting the steepest incline on which the train can be held with all wheels braked.

83. Two rings of equal weight are free to move on a rough horizontal rod, the coefficient of limiting friction being μ. They are connected by a smooth string of length l, on which another ring of the same weight as the other two rings together can move freely. Prove that, in a position of equilibrium of the system, the two rings on the rod cannot be further apart than

$$2\mu(1 + 4\mu^2)^{-\frac{1}{2}} l.$$

84. **PCQ** is a weightless string passing over a smooth peg at **C**; its ends **P,Q**, are attached to the extremities of a straight rod, not of uniform density. Prove that there cannot be more than one position of equilibrium in which the rod is not vertical; and determine whether such a position is one of stable or unstable equilibrium.

85. A uniform square lamina, of weight W and length of side $2a$, is placed in a vertical plane on two smooth pegs in the same horizontal line at a distance c apart. Prove that equilibrium may be secured in any position by properly weighting the lowest corner provided $c\sqrt{2} < a$, and that the necessary weight when a side is inclined to the vertical at an angle θ is

$$W\,\frac{a - c\,(\cos\theta + \sin\theta)}{c\,(\cos\theta + \sin\theta)}.$$

86. Four centres of force at the angles of a rectangle **ABA'B'** repel a particle **P** free to move along a circular wire circumscribing the rectangle, with the forces

$$\frac{\mu}{\mathbf{AP}},\ \frac{\mu}{\mathbf{BP}},\ \frac{\mu}{\mathbf{AP}},\ \frac{\mu}{\mathbf{BP}}$$

respectively. Prove that the particle can rest in equilibrium anywhere along the wire; and show that the resultant force is $\frac{\mu}{\mathbf{PM}} + \frac{\mu}{\mathbf{PN}}$, where **M**, **N**, are the feet of the perpendiculars from **A**, **B**, on the tangent to the circle at **P**.

87. Three equal uniform rods each of weight W have their ends hinged together so that they form an equilateral triangle; the triangle rests in a horizontal position with each rod in contact with a smooth cone of semi-vertical angle α, whose axis is vertical. Prove that the action at each of the hinges is $W \cot \alpha/\sqrt{3}$.

88. If equal lengths **OA, OB, OC** be taken along any three lines **Ox, Oy, Oz** in a plane, show that the resultant of forces $m \tan A$, $m \tan B$, $m \tan C$, acting along these lines, is the same as that of three forces acting along the same lines each equal to

$$m \tan A \tan B \tan C,$$

where A, B, C are the angles of the triangle **ABC**.

89. **ABCD** is a quadrilateral whose diagonals intersect at **E**, and whose centroid lies on the line bisecting the angle **AED**. Prove that if **L** is a point on **BD** such that **BL = ED**, and **M** a point on **AC** such that **AM = EC**, the triangle **ELM** is isosceles.

90. Three small particles are placed in a narrow smooth circular tube, and repel each other in such a way that the force between any two of them is proportional to the product of their masses and their distance apart. Show that when they are in equilibrium their masses are proportional to the sides of the triangle formed by drawing tangents to the tube at the positions of the particles.

91. If a system of forces in one plane reduces to a couple whose moment is G, and when each force is turned round its point of application through a right angle, reduces to a couple H, prove that when each force is turned through an angle α the system is equivalent to a couple whose moment is

$$G \cos \alpha + H \sin \alpha.$$

92. A uniform rod of length $2l$ is attached by smooth hinges at its ends to a vertical wall and to a uniform smooth sphere of the same weight as the rod, which rests against the wall. Prove that the cotangents of the angles which the rod, and the actions at the hinges, make with the vertical, are proportional to $3 : 2 : 4$. Prove also that if the rod is inclined to the vertical at an angle $\cot^{-1} 2$, the radius of the sphere is $l\sqrt{5}$.

93. A fixed rod **BD** makes an angle α with the horizontal; and a smooth weightless ring hangs by a string from the upper point **D**. A

uniform rod of length $2b$ passing through the ring rests with its lower extremity against **BD**. Show that, if the length of the string is $b\cot\alpha$, the rod is inclined to **BD** at an angle $\tan^{-1}(\cot\alpha)\frac{1}{3}$.

94. A cube of weight W rests on a smooth horizontal plane. A man wishing to get on to the top places a weightless ladder against it at an angle α to the horizontal, so that it just projects above the top of the cube, and he ties the foot of the ladder to the bottom of the cube. Prove that if $\alpha > 45°$, the man cannot reach the top of the ladder without the cube beginning to tilt, if his weight exceeds

$$W/[2\cos\alpha\,(\sin\alpha - \cos\alpha)].$$

95. A rhombus **ABCD** is formed of four equal weightless rods freely jointed at their extremities. The angular points **B, D** are joined by a string, and the system is strained by means of a second string connecting points in **AD, BC**, and perpendicular in direction to these sides. If T denote the tension in the latter string, and α the angle between the sides **AB, BC**, show that the tension in **BD** is

$$T\cos\alpha \,/\, \sin\frac{\alpha}{2}.$$

96. A triangle **ABC**, right-angled at **C**, formed of three uniform rods jointed at their ends, is suspended by a string attached to the middle point of **AB**. Show that the reaction at **C** is $cw \,/\, 2\sqrt{2}$, where w is the weight per unit length of each rod.

97. **ABCD** is a rhombus formed by light rods loosely jointed together: **OB, OD** are two equal rods jointed at **O, B**, and **D**. If **O** is connected to **A** and **C** by strings in tension, prove that the tension of each string is inversely proportional to its length.

98. A regular hexagon **ABCDEF** is made of six equal uniform rods jointed freely. The hexagon rests in a vertical plane having **AB** in contact with a given horizontal plane, and **C, F** connected by a light inextensible string. Show that the tension of the string is $W\sqrt{3}$, where W is the weight of a rod, and show that the action at the joint **E** is $W\sqrt{21}\,/\,6$.

99. Three rods of equal lengths are jointed together at their ends so as to form a triangle **ABC**; the middle points of the rods **D, E, F,**

are joined by stretched strings, the tensions of which are T, T, $2T$. Find the magnitudes and directions of the stresses at the joints, if $2T$ is the tension in **EF**.

100. Four rods freely jointed so as to form a quadrilateral rest on a smooth horizontal table under the action of forces at their middle points perpendicular to them. Prove that (1) a circle can be described about the quadrilateral, and (2) if the forces are proportional to the lengths of the rods on which they act, the reactions at the joints are all equal and are in the directions of the tangents to the circle at the corresponding points.

101. Nine equal weightless rods are freely jointed together at their ends so as to form the edges of a hexahedron, consisting of two regular tetrahedra upon the same base, and equal weights W are hung from four of the joints. If the whole hangs freely from the fifth joint, in which three rods meet, prove that the thrust along each of the three horizontal rods is $5W\sqrt{6}/18$.

102. A heavy uniform rod of length a is supported by two strings of lengths l, l', one attached to each end of it, the other ends of the strings being attached respectively to two points in a horizontal plane, their distance apart being b. Prove that the condition that the strings are at right angles is

$$(a^2 + b^2)(l^2 + l'^2) - (a^2 - b^2)^2 = \pm 4abll'.$$

103. Three equal weightless rods, **OA, OB, OC,** each of length c, are freely jointed at the fixed point **O,** and their ends **A, B, C** are respectively connected by three weightless inextensible strings, each of length l. The tetrahedron **OABC** rests with its face **ABC** horizontal and above **O.** A sphere of weight w and radius a is placed so as to rest symmetrically on the rods. Prove that the tension of the strings is $acw/l^2\sqrt{3}$.

104. Four equal uniform rods **AB, BC, CD, DE** are freely jointed at **B, C** and **D,** and are attached by smooth joints at **A** and **E** to fixed points in a horizontal line. If the length of each rod be a, and **AE** $= b$, show that if θ and ϕ be the inclinations of **AB** and **BC** to the vertical, then

$$\tan\phi = 3\tan\theta,$$

and

$$32a^2\sin^4\phi - 32ab\sin^3\phi + (8b^2 - 32a^2)\sin^2\phi + 36ab\sin\phi - 9b^2 = 0.$$

105. A chain **ABCDEF** is composed of five equal straight bars jointed together, and suspended from points **A, F** in the same horizontal line, whose distance apart is equal to the length of one of the bars multiplied by $2n + 1$. Show that the inclination of the bars **BC** and **DE** to the horizontal is determined by the equation

$$(\cos\theta - n)^2 (3\cos^2\theta - 4) + \cos^2\theta = 0.$$

106. **AB, BC, CD** are three equal and similar rods, freely jointed at **B** and **C**, and suspended from a point **A** by two strings **OA, OD**; the length of each string being equal to that of a rod, and each string being inclined at an angle α to the vertical, show that

$$36\sin^2\alpha = (4 + 5\sin^2\alpha)(1 - 2\sin\alpha)^2.$$

107. **ABCD** is a rhombus formed of freely jointed light rods; **EBFD** is another rhombus also formed of freely jointed light rods, lying within the first, and freely jointed to it at **B** and **D**. A weight W is suspended from **C, E** and **F** are connected by a string, and the whole is suspended from **A**. Find the tension in the string **EF**, given that the angle **ABC** $= 2\alpha$, and the angle **EBF** $= 2\beta$.

108. **ABCD** is a parallelogram formed of light rods freely jointed. The rod **BC** is produced to **F**, and **CF** $= n$. **BC**; also a point **E** is taken in **CD**, so that **CE** $= n$. **ED**. If **A, E** are joined by an elastic string in tension, and **E, F** by a freely hinged light rod, prove that the tension in **AE** is equal to n times the thrust in **EF**.

109. **ABCD** is a quadrilateral whose diagonals are equal, and whose sides **AD, BC** are equal. If **AC, CB, BD, DA** represent four freely hinged light rods, and **AB, CD** two elastic strings in tension, prove that in the equilibrium position the tensions in the strings are inversely proportional to their lengths.

110. If in the preceding problem **L, M, N** bisect **AD, AC, CB** respectively, and if instead of the elastic strings **DC** and **AB**, we have elastic strings **LM** and **MN**, prove that their tensions are again inversely proportional to their lengths.

111. A lever can turn about a fixed point at one end, and rests without friction against a fulcrum at $\frac{1}{n}$ of its length from that end, so as to be inclined at an angle α to the horizontal; show that when a

load P is suspended from the free end, which is so great that the weight of the lever itself is negligible, the lever will press against the pin of the joint with a force equal to

$$P\{(n-1)^2\cos^2\alpha + \sin^2\alpha\}^{\frac{1}{2}}$$

and give a construction for the direction of this force.

112. Prove that to transmit H horse-power by a shaft making N revolutions a minute, the turning moment must be $5250\,\frac{H}{N}$ ft. lb.

113. Determine the weight in tons which can be lifted by a rope wound on a capstan 2 ft. in diameter by 20 men, each weighing 12 stone, pushing horizontally against capstan bars 4 ft. from the ground, at a distance of 8 feet from the vertical axis of the capstan; the vertical through each man's C.G. overhanging his toes a distance of 1 foot.

114. A sculling boat is propelled by two sculls each 9 ft. long, and 6 ft. from rowlock to blade; and the sculler pulls each hand with a force of 20 lb. Prove that the thrust on each rowlock is 30 lb., but that the propulsive force on the boat is 20 lb.; and that the boat moves about double as fast as the hands pull. Estimate the H.P. at 8 miles an hour, assuming that the blades are in the water two-thirds of the whole time.

115. A circular ring, of radius a and weight W, is suspended in a horizontal position by three vertical threads, each of length l, attached to equidistant points on its circumference: prove that in order to hold the ring horizontal and twisted through an angle θ, a horizontal couple must be applied to it equal to

$$Wa^2\sin\theta/(l^2 - 4a^2\sin^2\theta)^{\frac{1}{2}}.$$

116. Determine the centre of gravity of a homogeneous triangular pyramid.

Show how the centre of gravity may be inferred of a heterogeneous triangular lamina, stratified in lines parallel to the base along which the density is proportional to the distance from (i) the base, (ii) the vertex.

ANSWERS.

Examples I. (Pages 20-23.)

1. About $15\frac{4}{5}$ lb. wt. 2. 16 lb. 3. 10 lb. and 15 lb.
4. (i.) 8**AP**, where **P** divides **BC**, so that $3\mathbf{BP} = 5\mathbf{CP}$.
   (ii.) 5**PA**, where **P** divides **BC**, so that $2\mathbf{BP} = 3\mathbf{CP}$.
   (iii.) 2**AP**, where **BC** is produced to **P**, and $2\mathbf{CP} = \mathbf{BC}$.
   (iv.) **PA**, where **BC** is produced to **P**, and $\mathbf{CP} = 4\mathbf{BC}$.
   (v.) 9**AP**, where **BC** is produced to **P**, and $9\mathbf{CP} = 2\mathbf{BC}$.
   (vi.) **PA**, where **BC** is produced to **P**, and $\mathbf{CP} = 8\mathbf{BC}$.
5. 13 lb. wt. 6. $\sqrt{3}P$, at right angles to $2P$ between $2P$ and $3P$.
7. See § 19; $2\sqrt{2}$ in a north-easterly direction. 8. 90°.
10. Zero. 11. In **P**, where $(m - k)\ \mathbf{BP} = (k - l)\ \mathbf{CP}$.
12. Draw **AF** parallel to **ED** to meet **BD** in **F**. 15. 83°.
16. Let **ABCD** be the quadrilateral, and let **F**, **G** bisect the sides **AB**, **CD**. Then the centre is the point bisecting **FG**, and the length of the radius represents one-fourth of the given resultant.
17. A line perpendicular to **AB**, dividing it in the ratio 1 : 3.
18. 60°. 24. (i.) $P : Q = \cos C : \cos B$. (ii.) $P : Q = \cos B : \cos C$.
31. 100 gms., 173 gms.
32. (i.) 50 lb., 37°. (ii.) 65 lb., $22\frac{1}{2}$°. (iii.) 5·83 tons, 31°.
   (iv.) 13·45 tons, 42°.

Examples II. (Pages 39-42.)

1. (i.) $2\sqrt{3}$ lb.; 2 lb. (ii.) 8 oz.; 8 oz.
   (iii.) 5 kilog.; $5\sqrt{3}$ kilog. (iv.) 0; 3 tons.
   (v.) -6 grs.; $6\sqrt{3}$ grs. (vi.) $-\frac{5}{2}\sqrt{2}$ lb.; $\frac{5}{2}\sqrt{2}$ lb.
   (vii.) $-4\sqrt{3}$ cwt.; 4 cwt. (viii.) -4 mgm.; 0.
   (ix.) 6 stone; 0.
2. $10\sqrt{5}$ lb.; inclined to vertical at an angle whose cosine is $2/\sqrt{5}$.
3. (i.) 1 lb. along each. (ii.) $\frac{3}{2}$ lb. along each.

4. (i.) 7 lb. (ii.) 26 grammes.
(iii.) 1 ton. (iv.) 5 kilog.
(v.) 5 cwt. (vi.) $2\sqrt{7}$ lb.
(vii.) $5\sqrt{3}$ lb. (viii.) $4\sqrt{10+3\sqrt{3}}$ mgm.
(ix.) $2\sqrt{13}$ $6\sqrt{3}$ oz.

5. $2\sqrt{39}$ lb.

6. 12 lb. opposite to the 10-lb. force, and $2\sqrt{3}$ lb. at right angles to this on the side remote from the 4-lb. force.

8. $2\sqrt{(35-18\sqrt{2})}$ lb. between north and west. **9.** $7\sqrt{3}$ lb.

10. $\sqrt{3}$ lb. **12.** If $P>Q$, Q and $\sqrt{(RS)}$ are perpendicular to each other.

13. $-\frac{5}{9}$. **14.** $-\frac{1}{7}$. **15.** $5(\sqrt{3}+1)$ lb., $5(\sqrt{3}-1)$ lb.

16. 16.93 lb., 10·65 lb.; 4·62 lb., ·78 lb.

18. (i.) 0°; (ii.) $22\frac{1}{2}$°; (iii.) 0°; (iv.) 8°; (v.) 8°; (vi.) 19°; (vii.) 30°; (viii.) $7\frac{1}{2}$°; (ix.) 38°. **19.** 132°. **20.** 2·77 lb.

21. $a^2(b^2+c^2-a^2) : b^2(c^2+a^2-b^2) : c^2(a^2+b^2-c^2)$.

22. $P = R\sin\beta / \sin(\alpha+\beta)$; $Q = R\sin\alpha / \sin(\alpha+\beta)$. **23.** $\frac{1}{5}P\sqrt{5}$.

24. The cosines of the angles are $-\frac{7}{8}$, $-\frac{11}{16}$, $\frac{1}{4}$. Hence the angles are 151°, 134°, 75°. **30.** 2 : 1.

Examination Paper I. (Page 43.)

1. See § 5. **2.** See §§ 12, 13. **3.** See § 16.

4. $5\frac{\mathbf{AP}}{\mathbf{AB}}$ lb. in direction parallel to **AP**, where **BC** is produced to **P**, and $2\mathbf{BC} = 5\mathbf{CP}$.

5. See §§ 18, 20, 21. **6.** See § 23. **7.** See §§ 27, 28.

8. Draw **DF** parallel to **EA** to meet **AB** in **F**.

9. $R^2 = P^2+Q^2+2PX$, where $X = Q\cos(P, Q)$. **10.** See § 34.

Examples III. (Pages 62-66.)

1. (i.) $7\frac{1}{2}$ lb.; 6 lb. (ii.) $32\frac{1}{2}$ lb.; 30 lb. (iii.) $8\frac{3}{4}$ tons; $8\frac{2}{5}$ tons. (iv.) $45\frac{1}{3}$ kilog.; 40 kilog.

2. (i.) $\frac{5}{3}\sqrt{3}$ tons; $2\frac{1}{2}$ tons. (ii.) 28 lb.; $14\sqrt{2}$ lb. (iii.) $10\sqrt{3}$ kilog; $5\sqrt{3}$ kilog. The reactions are respectively (i.) $\frac{10}{8}\sqrt{3}$ tons; $\frac{5}{2}\sqrt{3}$ tons; (ii.) $28\sqrt{2}$ lb; $14\sqrt{2}$ lb.; (iii.) 20 kilog.; 5 kilog.

3. (i.) $6\frac{3}{4}$ lb. (ii.) $5\frac{2}{5}$ lb. **4.** $6\sqrt{2}$ lb.

5. $\frac{10}{3}\sqrt{3}$ lb. **7.** 75 lb. **8.** 2·5 lb.

9. At right angles to the first string. **10.** 5 lb.; $\frac{5}{2}\sqrt{3}$ lb.

11. Along the plane, $W \sin \alpha$; the force must act towards the plane at an angle of $(90° - \alpha)$ with it.

12. $T = \frac{W}{2} = \frac{W'\sqrt{3}}{2}$. 13. $\frac{Q^2}{P}$.

14. $28\sqrt{2}$ lb.; when the angle between the two parts of the string is equal to or greater than 120°.

15. Anywhere in the tube.

17. The tension in each string equals any one of the weights.

20. 30°. 22. 2 ft. 2 in.

23. $W \sin \alpha / \sin(\alpha + \beta)$; $W \sin \beta / \sin(\alpha + \beta)$.

24. 30°, 45°, 60°. 25. $\frac{1}{8}P\sqrt{3}$. 26. $\{3p^2q^2/(q^2 - 4p^2)\}^{\frac{1}{2}}$.

27. 1 lb. wt. 28. $\frac{3}{4}$. 29. 5 : 3 : 4; $2P$.

30. 30° to the horizontal; 30° above the plane.

31. 30° to the horizontal; 45° above the plane. 32. $W\sqrt{3}$; W.

33. Tension of vertical string $= W$; of the other, $2W$.

34. 25 : 52. 35. 3·46 lb. 36. 6 lb., 8 lb.

38. Opposite to the first force, and 60° from the first force.

Examples IV. (Pages 81-85.)

1. See § 58. 2. $\frac{3}{5}$ lb.

5. The rod is inclined at 30° to the vertical; 32 lb.

6. $1\frac{1}{2}$ in.; 60° with horizontal.

7. $\frac{\sqrt{7}}{4}W$, $\frac{\sqrt{3}}{4}W$, where W is the weight of the rod.

8. $10\sqrt{2}$ lb.; 10 lb. 9. $\frac{\sqrt{3}}{2}W$; $\frac{W}{2}$. 11. $\frac{\sqrt{3}}{2}$ lb.; $\frac{1}{2}$ lb.

12. $4a$. 13. Stress at **B** $= pW/q$; stress at **C** $= W\sqrt{p^2 + q^2}/q$.

14. $W(l + r)/\sqrt{l^2 + 2lr}$. 15. $\frac{1}{3}W\sqrt{6}$. 16. 1·25 lb. wt.

19. Bisect **AB** at **G**; let the vertical through **G** meet the horizontal through **A** at the point **D**; **BD** produced meets the wall in the required point **C**.

22. $\tan^{-1}\frac{1}{3}$. 27. $\frac{Wr}{\sqrt{a^2 + b^2}}\left\{\frac{a}{\sqrt{4r^2 - a^2 - b^2}} + \frac{b}{\sqrt{a^2 + b^2}}\right\}$.

29. $r(\sec \alpha - 1)$. 30. $-r(\sec 2\alpha + 1)$.

Examination Paper II. (Page 86.)

1. See § 52. 2. See § 53. 3. See § 42.

4. See § 45. 5. 15 lb. 6. See § 46.

7. $2\sqrt{3}$ lb., at right angles to the 5-lb. force between the 5-lb. and 7-lb. forces.

8. See § 53. The line through **C** perpendicular to **AB**, the horizontal line through **A**, and the vertical line bisecting **AB** must all meet in a point. 9. $Q = bP/a$, $R = cP/a$.

EXAMPLES V. (PAGES 99-103.)

1. (i.) $2\frac{1}{2}$, (ii.) $\frac{5}{4}\sqrt{2}$. (iii.) $\frac{5}{4}\sqrt{2}$, (iv.) $\frac{5}{4}\sqrt{3}$, in ft.-lb. units.
2. $\frac{3}{2}$**AC**; 4**AC**; $\frac{5}{2}$**AB**.
3. Towards the side on which **A** lies. 4. See §§ 60, 68; $\sqrt{3} : 4$.
6. About one end, 0, + 4, + 18, − 32; about other end, + 8, − 12, − 6, 0; about middle point, + 4, − 4, + 6, − 16. The sum of the moments is − 10 round any one of the points.
7. 0, $\frac{3\sqrt{3}}{2}$ **AB**, $5\sqrt{3}$**AB**, $7\sqrt{3}$**AB**, $\frac{9\sqrt{3}}{2}$ **AB**, 0.
8. $8\sqrt{3}$, $10\sqrt{3}$; at **P** in **BC** produced, such that **CP** = 16 ft.
9. $(P - Q)a / \sqrt{P^2 + Q^2}$, where a is the length of the side.
10. Two straight lines parallel to the resultant at a distance of $\frac{1}{4}$ the unit length on each side of it.
11. The resultant passes through **A**.
12. The moments and sums of the moments round the four corners are respectively
    (i.) − 3, − 4, + 10, + 12; + 15. (ii.) + 6, − 4, − 5, + 12; + 9. (iii.) + 6, + 8, − 5, − 6; + 3. (iv.) − 3, + 8, + 10, − 6; + 9.
13. Any point in a line parallel to **CD** and distant from it $2a$ on the side opposite to **AB**, where a is the length of a side of the square.
15. $45\sqrt{3}$ lb. wt. 16. $V + \frac{1}{2}W$.
17. 50 lb. wt.; $25\sqrt{2}$ lb. wt.
25. Divide **AB** at **C** in the ratio $\mu' : \mu$; the locus of P is a line parallel to **OC**. 28. 30 inches.

EXAMPLES VI. (PAGES 117, 118.)

1. (i.) 4 lb.; $1\frac{1}{2}$ ft.,* $\frac{1}{2}$ ft. (ii.) 12 lb.; 21 in., 15 in.
   (iii.) 12 lb.; $31\frac{1}{2}$ in., $4\frac{1}{2}$ in. (iv.) 14 lb.; 30 in., 12 in.
   (v.) $\frac{3}{4}$ ton; 4 in., 2 in. (vi.) 1 kilog.; 6 cm., 4 cm.
2. (i.) 2 lb.; 3 ft., 1 ft. (ii.) 2 lb.; $10\frac{1}{2}$ ft., $7\frac{1}{2}$ ft.
   (iii.) 9 lb.; $3\frac{1}{2}$ ft., $\frac{1}{2}$ ft. (iv.) 6 lb.; 70 in., 28 in.
   (v.) $\frac{1}{4}$ ton; 12 in., 6 in. (vi.) 200 gm.; 30 cm., 20 cm.

* In Examples 1, 2, the distance of the resultant from the smaller component is given first.

3. $5\frac{15}{19}$ ft. from 8-lb. end ; 19 lb.
4. 1 unit, at a distance of 1 ft. from the force of 2 units, and 2 ft. from the force of 1 unit.
5. The bar balances about a point distant $1\frac{1}{4}$ ft. from the boy.
6. See § 78. 7. $68\frac{2}{11}$ lb., $81\frac{9}{11}$ lb. 8. 25 lb.
9. $P = 8$ lb., $Q = 9$ lb. 10. 12 lb.
13. 6 lb. ; 8 ft. from the table. 14. $(P^2 - Q^2)/P$, if P be the greater.
15. Force at **B** is P sin **B** cos **C** cosec **A**.
16. Force at **B** is P sin $2\mathbf{B}$ / (sin $2\mathbf{B}$ + sin $2\mathbf{C}$).
17. Produce this side backwards through a distance equal to twice the side of the square.
18. $l - a(W - P) / W$.

Examples VII. (Pages 131-133.)

1. 3 ft. from the man carrying 71 lb.
2. 21 lb., acting between the 7-lb. and 9-lb. forces at a distance of 3 ft. from the latter.
3. When the 100 lb. is placed on the end furthest from a prop, the pressures are 225 lb. downwards on the nearer prop, and 5 lb. upwards on the further prop ; when the 100 lb. is placed on the other end, the downward pressures are 170 lb. on the nearer and 50 lb. on the further prop.
4. $17\frac{1}{2}$ lb.
5. 11 lb. acting in the same direction as the 7-lb. force, at a distance of $\frac{3}{11}$ ft. from the end where the 3 lb. is.
6. $3\frac{1}{2}$ cwt. on each. 7. 3 ft. from the first prop. 8. 228.
9. (*a*) $36\frac{2}{3}$ lb., $54\frac{4}{9}$ lb. (*b*) $78\frac{4}{9}$ lb., $12\frac{8}{9}$ lb. (*c*) $-5\frac{3}{5}$ lb., $96\frac{3}{5}$ lb.
10. 540.
11. If P is the force, and M the moment of the couple, resultant is equal and parallel to P, and at a distance M/P to its right.
12. 15 units parallel to **AC**, and at a distance from it equal to $\frac{4}{3}$**AB**.
13. See § 94. 14. See § 93.
15. (*a*) $\frac{2P}{\mathbf{BC}}$, area of triangle **ABC**.

    (*b*) $2P$ acting along the line bisecting **AC** and **BC**.
16. Complete parallelogram **ABDC**. Required force is equal and parallel to **DA**, at a distance from it $= \frac{1}{4}$ the distance of **B** or **C**.
17. $2\pi nM$; $942\frac{8}{7}$ ft.-lb.
21. sin $2A$: sin $2B$: sin $2C$.

Examination Paper III. (Page 134.)

1. See § 61. 2. See § 69, and converse of § 80, Alternative Proof.
3. See § 78.
4. (*a*) 300 ft.-lb. units ; 750 ft.-lb. units, in opposite senses.
   (*b*) 450 ft.-lb. units ; to the side of **AC** on which **B** lies.
5. See § 81. 6. 21 lb. on **A**, 9 lb. on **B** ; 10 lb. more in each case.
8. 4 lb. wt., acting parallel to, and in the same direction as, the given force of 4 lb., and at a distance of $1\frac{1}{2}$ ft. to its right.
9. See §§ 86, 93. 10. See § 95, Cor.

Examples VIII. (Pages 156-159.)

1. 15 lb. 2. $\frac{2}{3}$ cwt., acting downwards. 3. 8 lb., 10 lb.
5. 3 ft. from the man bearing 94 lb.
6. 10 lb., 1047 $\frac{13}{21}$ ft.-lb. 8. $94\frac{1}{2}$ lb. 9. $4\frac{1}{2}$ lb.
10. $P = 9$ lb. ; $Q = 15$ lb. 12. $\frac{1}{3}$**AB** from **A** ; pressure $= P\sqrt{3}$.
13. Tension of thread $= W$; pressure at **C** $= W\sqrt{3}$ perpendicular to **AB**.
14. 3 lb. ; 3 ft.
15. Radius of wheel is four times radius of axle ; weight of wheel and axle and weight of man. 16. See § 111.
17. Replace R by its value $P + Q$, and combine the two terms with the effort and weight, respectively.
19. $P : Q = b \sin \beta : a \sin \alpha$;
$$\text{reaction} = \sqrt{[P^2 + Q^2 - 2PQ \,.\, \cos(\alpha + \beta)]}.$$
20. $(b^2 + a^2)\ (b - a)\ W\ /\ (2a^3)$. 21. 60°.
22. $2W \sin \alpha \,.\, \text{cosec}\ (\alpha + \beta)$.
23. 65 lb. wt. ; $\tan^{-1} 1{\cdot}2$ below the horizontal.
24. It makes an angle $\tan^{-1} \frac{8}{7}$ with the downward vertical.
25. $\frac{1}{2}W(a - b)\ /\ \sqrt{[l^2 - (a - b)^2]}$.
26. (*a*) 60 lb. wt. ; (*b*) 20 lb. wt. ; (*c*) 40 lb. wt. Because the reactions of the sculls on the hands pull the man backwards.
27. $\frac{1}{2}$ lb. wt. 28. ·15.

Examples IX. (Pages 179-182.)

1. $\frac{1}{16}$, 20 lb. 2. $\frac{1}{31}$, 14 lb. 3. 6 cwt. 4. Three moveable pulleys.
5. 22 lb. 6. Light in the first system, heavy in the third.
7. 7 lb. 8. 8 lb., 4 lb., 15 lb. 9. 16 lb., 5 lb., 7 lb.
10. 250 lb. ; 250 lb., 500 lb., 1000 lb., 2000 lb. ; difference $= P$.
11. $7\frac{1}{3}$ stone, $18\frac{2}{3}$ stone. 14. 9 stone.

16. The radii should be as the numbers 1, 2, 3, . . . , or as 1, 3, 5 . . . , according as the string is fastened to the lower or to the upper block.
17. Between the strings over the fixed pulley and the next, and $\frac{11}{15}$ ft. from the former.
18. 1508 lb. nearly. 19. $\frac{33}{49}$ in. 20. $\frac{490}{297}$ lb. 21. 224.
22. $\frac{1}{2}Wl \,/\, \sqrt{(l^2 - d^2\cos^2\alpha)}$. 23. $g/4$; $g/24$. 24. $g/3$; $g/19$.
25. $g(Q - P)/P$; $g(PQ - P^2)/(P^2 + QW)$. 26. $2\pi r/(a - b)$.
29. $W = P(2^{2n+1} + 2^n) - w(2^{2n+1} - 3.2^n + n + 2)$.

Examination Paper IV. (Page 183.)

1. See §§ 104-106. 2. See § 105. 3. See §§ 111, 112. 4. 9 lb.
5. See § 128. 6. θ. 7. $3\frac{57}{245}$ in.; $\frac{18}{35}$ in.
8. 3 lb., mechanical advantage $= 6$. 9. $\frac{1}{2}W\cos\alpha \,.\, \operatorname{cosec}(\alpha + \beta)$.

Examples X. (Pages 196, 197.)

1. (i.) The centre of the triangle. (ii.) The vertex of the equilateral triangle described on the other side of the base opposite the point of application of the unlike force.
2. Take **F** in **AB**, so that $2\mathbf{AF} = \mathbf{FB}$; the centre of the forces is at **G** in **FC**, where $9\mathbf{FG} = 2\mathbf{GC}$.
3. Produce **BA** to **F**, so that $\mathbf{AF} = \mathbf{AB}$; the centre of the forces is at **G** in **FC**, where $3\mathbf{FG} = 2\mathbf{GC}$.
4. 4 in. 5. $\frac{2}{5}$ in. nearer the centre.
6. It is moved from the centre of the square to the corner opposite the point of application of the reversed force.
7. At an infinite distance. 8. $1\frac{1}{12}$ in. from the 3 oz. mass.
9. 12 : 7 : 2; 2 : 1, 6 : 1.
10. 7 oz.; $9\frac{1}{7}$ in. from **A**. 11. See § 150.
12. (i.) $P : Q : R = a : b : c$; (ii.) $P : Q : R = -a : b : c$;
(iii.) $P : Q : R = \sin 2A : \sin 2B : \sin 2C$;
(iv.) $P : Q : R = \tan A : \tan B : \tan C$.

Examples XI. (Pages 210-212.)

1. 6 lb. 2. See § 153, Ex. 3. 12 lb.; at middle point of rod.
4. 2 oz.; 12 in. 5. See § 153.
6. See § 156; if the 8-in. side was originally vertical, the block will topple when the plank forms an inclined plane, such that height $= \frac{5}{8}$ base.

7. It trisects the straight line bisecting **AB** and **CD**.
8. The sphere containing the lead has *two* positions of equilibrium only when placed on a table—one stable, the other unstable; the other sphere is neutral in all positions.
10. 450 sq. ft. **11.** 50 lb.; 1 ft. from the centre.
15. (i.) Yes; (ii.) unstable; (iii.) $\frac{2}{3}$ lb. **17.** 40 inches.

Examples XII. (Pages 228-232.)

1. Suspend it by *three* strings from one point from which is hung a plumb-line. This cuts the plane of the hoop in the required C.G.
2. 2 ft. from vertex; 8 in. from vertex.
3. $\frac{13}{22}$**AB**, $\frac{8}{11}$**AB**, from **AB** and **AD** respectively.
4. 4 in. from **AB**, $3\frac{1}{2}$ in. from **AD**. 5. $\frac{1}{12}$**AC** from **O** towards **C**.
7. $\frac{2}{15}a$ from **O** towards middle point of **DE**. 8. See § 177.
11. $\frac{1}{6}$**AB** from **O** towards middle point of **CD**. 12 See § 169.
13. 4·32 in. from base.
14. See § 178. The volumes of the whole cone and the part cut off are proportional to the cubes of their axes; hence apply § 173.
15. 1 in. and $1\frac{1}{2}$ in.
16. $\frac{Wl}{2}$, where W is the weight and l the length of the chain.
17. One-sixth of radius from centre of board.
20. The masses are proportional to **BC**, **CA**, and **AB**. 22. $\frac{39+4\sqrt{6}}{75}a$.
23. One-third of the diagonal from the point where the weight is suspended.
24. At the middle point of the common side.
25. 16 ft.-lb.; $5\frac{1}{3}$ in. **26.** 192,500,000 ft.-lb. **28.** $\frac{3}{2}a$.
29. The angles between the radii which join the particles to the centre must be 135°, 135°, 90°. **31.** 2 in. from **A**.
32. 55 lb. wt. on the front wheel; 72·5 lb. wt. on each back wheel.
35. $11\frac{1}{3}$ in. from **A**; $13\frac{2}{3}$ in. from **C**. **36.** 6 lb.
39. 10·9 in. **40.** $6r^2/\sqrt{(9r^2+h^2)}$.

Examination Paper V. (Page 232.)

2. See § 150. 3. $7\frac{5}{16}$ in. from **A**; 46 in. from **A** in **BA** produced.
4. 2 in. from centre of box towards the middle point of the vertical face to which the hinges are attached.

5. See § 176. 6. Same as C.G. of original triangle.
8. $2\frac{11}{15}$ in.

Examples XIII. (Pages 244, 245.)

1. See § 190. 2. $\frac{141}{146}$ lb. 3. 32 in. from fulcrum.
4. Its C.G. is at the fulcrum. 5. No.
6. 5 in. from the weight. 7. 15 lb., 26 lb.; at middle of rod.
8. 1 lb., 2 lb., 4 lb., 8 lb., 16 lb.; 1 lb., 3 lb., 9 lb., 27 lb.
9. With diminished weight he cheats his customers, with increased weight he cheats himself.
10. When the weight is hung by a string. 12. 10 stone.

Examination Paper VI. (Page 246.)

1. See § 182.
2. See §§ 191, 192; only one weight used, and this a comparatively small one.
3. See §§ 194, 195. 4. See § 186. 5. 10 lb. $\frac{1}{2}$ oz.
6. See § 175. 7. 20,000 ft.-lb.
8. If **ACB** be the thread and **C** the position of the weight, **AC** = 6 ft., **CB** = 3 ft.
9. $\frac{1}{5}a$ from centre of hexagon to corner opposite to that at which no weight is situated, a being side of hexagon.

Examples XIV. (Pages 256, 257.)

6. Intersection of medians. 8. Yes. 9. 21 ft.-lb. units.
10. **DB** must bisect **AC**; 2**BD** along **BD**, 0 along **AC**.
12. Sum of moments about the opposite angular point must be zero.

Examples XV. (Pages 273-280.)

1. 1 : 3. 2. $3aW/2\sqrt{(l^2 - 4a^2)}$. 9. W.
10. $\frac{1}{4}W\sqrt{3}$; $\frac{1}{2}W$, inclined at 30° to the horizontal.
12. $\sin^{-1}[(W + 2w)\, c/2aw]^{\frac{1}{3}}$.
15. Upper pair, $3W \sec \alpha/2$; lower pair, $\frac{1}{2}W \sec \alpha$; horizontal rod, $2W \tan \alpha$.
18. $\frac{1}{2}W \cot \alpha$, horizontal. 20. $\frac{1}{8}(W + w)\sqrt{3}$.
21. $5W\sqrt{3}/2$ along **BF**; $W\sqrt{3}/2$ along **CE**.
22. $2W$; W, vertical. 23. $25W\sqrt{2}/24$.
24. $3W/\sqrt{5}$; $W\sqrt{10}/5$ along **BC**.

27. $W\sqrt{21}/6$; $\tan^{-1}\frac{\sqrt{3}}{2}$ from horizontal.

28. Upper hinge, 160 lb. vertical, $293\frac{1}{3}$ lb. horizontal ; lower hinge, $293\frac{1}{3}$ lb. horizontal.

29. $\sqrt{3} : 2$. **30.** $7\frac{5}{13}$, $9\frac{3}{13}$, $5\frac{7}{13}$ lb. **31.** $\frac{1}{2}W(\sqrt{3} - 1)$.

35. $F \cos \mathbf{C} \operatorname{cosec} \mathbf{A}$; $F \cos \mathbf{B} \cos \mathbf{C} \operatorname{cosec} \mathbf{A}$. **40.** 1, 1, ·7, ·7, ·37 lb.

41. 7·4, 2·3 lb. (more accurately, 7·35, 2·27 lb.).

42. 7·4 (more accurately, 7·38), 1·3 tons. **45.** $2\frac{3}{4}$, $2\frac{3}{4}$ inches.

Examples XVI. (Pages 293, 294.)

3. 30°. **8.** $2W$. **10.** $\frac{1}{2}T(\sqrt{2} + 1)$. **11.** $W \tan \alpha$.

15. 30°.

Examination Paper VII. (Page 295.)

8. $a/(l - a)$. **9.** 60°.

Examples XVII. (Pages 307-311.)

2. 24. **3.** $a(2\pi r - b\mu)/r(\pi r\mu + b)$. **4.** 4. **5.** $4\sqrt{3}/3$.

6. 20. **7.** 1. **11.** $5\sqrt{3}/36$; $2\sqrt{3}/5$. **12.** $\cdot\frac{3}{4}$. **16.** 2θ.

20. 8 lb., 4 lb **21.** ·25 ; 6·. **25.** $2 \tan^{-1} \mu$.

26. (1) The inclination of the beams to each other must not be greater than $2 \tan^{-1} 2\mu$; (2) Not less than $lW(\tan \alpha - 2\mu)/2d$, where 2α is the angle between the beams, l the length of each beam, d the distance of the point of attachment of the rope from the top, and W the weight of each beam.

28. W . $\sin(\alpha + \gamma)$. $\sec(\theta - \gamma)$; $W \sin(\alpha - \gamma)$. $\sec(\theta + \gamma)$; where γ is the angle of friction.

29. (1) $W \sin(\alpha + \gamma)$; (2) $W \sin(\alpha - \gamma)$; where α is the inclination of the plane and γ the angle of friction.

32. Twice the angle of friction.

Examples XVIII. (Pages 325-328.)

2. Distance from centre $= \frac{45}{112}r$.

11. Distance of C.G. from centre $= r(27 - 4\pi) / 3\pi$.

17. The line joining the extremities is inclined to the horizontal at the angle $\sin^{-1}(\frac{1}{2}\mu\pi)$. **18.** $\sin^{-1}(\frac{1}{4}\pi \sin \alpha)$.

19. Height of cone $= 4 \times$ radius of sphere. **20.** (i.) $\frac{3}{5}r$; (ii.) $\frac{6}{5}r$.

21. $2r$. **24.** $\angle \mathbf{BAC} = 60°$. **25.** The vertical position.

Examination Paper VIII. (Page 329.)

3. W and $\frac{1}{3}\sqrt{3}\,.\,W$.
5. $W\mu\,/\,\sqrt{(1+\mu^2)}$ in a direction inclined to the plane at the angle of friction. 8. At a distance $3r\,/\,8$ from the centre.

Miscellaneous Exercises. (Pages 330-346.)

2. Tension $= W$; thrusts, W and $\sqrt{2}\,.\,W$.
3. $3\frac{1}{2}$ cwt. and $3\frac{1}{2}$ cwt. 6. $W\,2\sqrt{3}$.
7. $W\,/\,\sqrt{3}$. 8. $2W$. 9. $2W\sqrt{3}/9$. 10. 2 : 1.
12. $W\sqrt{3}$. 13. $W\sqrt{3}$. 14. 36 : 64. 15. 50.
16. W; 0; ·75 W; 1·25 W. 17. $3P/\sqrt{10}$. 20. 1 cwt.
21. $\frac{1}{3}W\sqrt{3}$. 25. Slide. 35. Unstable.
36. W cot B cot C. 37. W cos $A/2$ sin B sin C.
38. $5W/6$; $2W\sqrt{5/3}$. 43. μr W cosec $\alpha\,/\,(\mu^2+1)$.
44. 30° from horizontal. 46. ·366.
47. $W(1+m)^3\,/\,(3+3m+m^2)$; 8483 about.
48. $P:Q = R\sqrt{1+\mu^2} - r\mu : R\sqrt{1+\mu^2} + r\mu$; $\mu =$ ·1 nearly.
49. $\frac{1}{3}W(\sqrt{3}-\sqrt{2})$. 50. ·134$W$.
52. The orthocentre of the triangle.
56. Distance of c.g. from the vertex is $(\rho + 7\rho')\,/\,3(\rho+3\rho')$ of the median.
66. $Wa\,/\,\sqrt{(c^2-2ac)}$.
67. Each pressure between the rod and the cylinder is $\frac{1}{2}$ cot α × weight of rod; pressure on the plane = weight of rod.
69. $W\sqrt{2}/8$.
76. If W_1, W_2 are the weights of **AB**, **BC**, the action at the hinge is vertical and equal to $\frac{1}{2}W$; the pressure on the prop is $\frac{1}{2}W_1 + W_2$; the prop must be placed at **D**, where
$$\mathbf{BD} = 2W\,.\,\mathbf{BC}/(2W_2 + W_1).$$
77. **PC** . sin 2**PCA**/sin $2A$.
78. $r \pm 2k\Delta/\mathbf{P}$, where Δ is the area of the triangle.
84. Unstable.
99. Reaction at **A** is $\frac{3}{2}T$, parallel to **FE**; reactions at **B** and **C** are each $\frac{1}{2}T\sqrt{7}$, and their lines of action are inclined to **AB**, **AC** respectively at the angle $\tan^{-1}3\sqrt{3}$.
107. W cot α tan β. 113. 1·5 tons. 114. ·28.
116. (i.) c.g. bisects the median; (ii.) c.g. divides the median from the vertex in the ratio 3 : 1.

EXAMINATION QUESTIONS.

(LONDON UNIVERSITY.)

Intermediate Science.

1. Find the magnitude, direction, and line of action of the resultant of three forces of 1, 2, and 3 units acting in order round the three sides of an equilateral triangle of side $2a$.

2. State necessary and sufficient conditions for the equilibrium of a system of coplanar forces.

A rod of length a and weight w is supported in a horizontal position by two strings of length a attached to the ends of the rod and to two rings of weight $\frac{1}{4}w$ which are free to move along another fixed rough horizontal rod. If μ is the coefficient of friction between the rings and the rod, show that the greatest distance apart of the rings consistent with equilibrium is $a\left\{1 + \frac{6\mu}{\sqrt{9\mu^2 + 4}}\right\}$.

3. Show that the work done in raising any number of masses through any vertical distances is equal to the work done in raising a mass equal to the sum of the masses through the vertical distance the centre of gravity of the masses is raised.

A rectangular box of edges a, b, and c and weight w is lying on a face with the edge a vertical. It is turned about an edge until the edge b is vertical and then about an edge until the edge c is vertical. Find (i.) all the work done against gravity, neglecting the work done by gravity, (ii.) the net work done by or against gravity.

4. Describe a differential pulley block or a differential wheel and axle and calculate the velocity ratio of the machine described.

Explain how in practice the efficiency of the machine is determined.

5. Prove that if three forces are in equilibrium the sum of their moments about any point is zero.

A gate hangs from two hinges at A, B, its centre of gravity being G, and the vertical through G meets the horizontal through A in K. Prove that if the whole weight is supported at B the triangle ABK will serve as a triangle of forces to determine the reactions at A, B; and modify this construction when the weight is equally divided between A and B.

6. Find the centre of mass of a triangular area.

$ABCD$ is a quadrilateral in which AB, CD are horizontal and of lengths a, b: prove that the depth of the centre of mass below AB is

$$\tfrac{1}{3}h\,(a + 2b)/(a + b),$$

where h is the depth of CD below AB.

7. Two equal uniform rods AB, BC, smoothly jointed together at B, are in equilibrium with the end C resting on a rough horizontal plane and the end A held above the plane. Prove that, if α and β are the inclinations of CB and BA to the horizontal, the coefficient of friction must exceed

$$2 \div (\tan\beta - 3\tan\alpha).$$

8. A train of weight W and length l is at the centre of a bridge of twice its own length. Assuming the weight of the train uniformly distributed throughout its length, calculate the bending moment at the centre of the bridge, and compare it with the value when one end of the train just reaches one of the piers of the bridge.

1. If forces 1, 2, 3 act along sides AB, BC, CA respectively: produce BC to D, making $CD = a$. The resultant is a force $\sqrt{3}$ units acting at D, at right angles to BC.

3. $\frac{1}{2}w\{\sqrt{(a^2+b^2)}+\sqrt{(b^2+c^2)}-a-b\}$; $\frac{1}{2}w(c-a)$.
5. Triangle of forces is the isosceles triangle on base AB whose vertex lies in GK.
8. $\frac{3}{8}Wl$; ratio 3 : 2.

Intermediate Arts.

1. Show that the moment of any number of forces in a plane about a point O of the plane is equal to the moment of their resultant about O.

The moments of a system of forces about two points A, B of their plane are equal. Show that the forces either reduce to a couple or have their resultant parallel to AB.

2. Prove that, if x be the coordinate (parallel to any axis) of the centre of gravity of a number of bodies of masses $m_1, m_2, \ldots m_n$, the coordinates of whose centres of gravity are $x_1, x_2, \ldots x_n$ respectively, then

$$x = \frac{m_1x_1 + m_2x_2 + \ldots m_nx_n}{m_1 + m_2 + \ldots + m_n}.$$

Find the centre of gravity of an L-shaped lamina of uniform thickness and density, the outside height and breadth of the vertical arm being 10 inches and 1 inch, and the outside length and breadth of the horizontal arm 6 and 2 inches respectively.

3. Prove that if the fulcrum of a balance be not immediately above the centre of gravity, but the arms are equal, a correct result is obtained by weighing the body alternately in the two pans and taking the arithmetic mean of the results.

Show further that if, in addition, the arms be unequal and differ by an amount h, the total span being l, then, to obtain a correct result, the arithmetic mean above must be diminished by

$$\frac{1}{2}(W_1 \sim W_2)\frac{h}{l},$$

where W_1, W_2 are the apparent weights in the two cases.

4. A smooth uniform ladder rests with its extremities against a vertical wall and a horizontal plane respectively, and is held by a rope, one end of which is attached to a rung of the ladder one-fourth of the way up, the other end being fixed to a point of the base of the wall vertically below the top of the ladder.

Show that if the base and top of the ladder be distant a feet and b feet respectively from the base of the wall, the ratio of the reactions P and Q between the ladder and the ground and wall respectively is given by

$$Q : P = 3a/5b.$$

5. Four light rods AB, CD, BC, AD of lengths 4, 10, 5, and 5 feet respectively, freely jointed at A, B, C, D, form a trapezium in a vertical plane, AB and CD being horizontal and AB above CD. The whole is placed on smooth horizontal supports at C and D and a weight of 200 lb. is placed at the middle point E of AB. Find the stresses in the rods AD, BC, CD.

6. State and prove the triangle of forces.

A picture of weight 5 lb. is hung from a nail by a cord 5 ft. long fastened to two rings 3 ft. apart ; find the tension in the cord graphically or by calculation.

7. An equilateral triangle ABC, formed of three light rods each 2 ft. long joined at their ends, is supported at a point in AC distant 15 in. from A, and a weight of 13 lb. is hung from a point in AB distant 7 in. from A. Find (by a graphical or other method) the tension in the rod BC and the action between the other two rods at A. (A numerical result is required.)

8. Define the centre of gravity of a body : show how to find the centre of gravity of a body made up of two parts whose masses and centres of gravity are known.

A circular disc of radius R and thickness T lies on a table ; a second disc of the same material of radius r and thickness t is laid on the first, the distance between the centres of the faces in contact being x. Find the distance

of the centre of gravity of the two discs from the table and from the axis of the lower disc.

2. The point (2, 3), taking the outside edges as axes of coordinates.
5. 125 lb., 25 lb., 75 lb. 6. $3\frac{1}{8}$ lb.-wt. 7. 4·38 lb., 9·6 lb.
8. $\{R^2T^2 + r^2(t^2 + 2tT)\}/2(R^2T + r^2t)$; $r^2tx/(R^2T + r^2t)$.

Intermediate Science.

1. Investigate the necessary and sufficient conditions that a system of coplanar forces, acting on a rigid body, may be in equilibrium.

Four equal uniform rods are freely jointed to form a rhombus $ABCD$. The rhombus is suspended from the joint A, and is maintained in the form of a square by means of a rod of negligible weight joining the middle points of BC and CD. Find the vertical and horizontal components of the reactions at the joints B and D.

2. Explain the terms "couple," "moment of a couple."

Two equal rods AB, BC are smoothly jointed at B ; the end A is free to turn round a smooth pin, while C is free to slide along a smooth slot whose direction passes through the pin A. Show that if a force P is applied to the end C along the slot, the two rods can be kept in equilibrium by a couple applied to AB, whose moment is $2Pp$, where p is the perpendicular from B on AC.

3. A wedge, of weight W, with two plane faces inclined at an angle α, is placed with one of these faces in contact with a horizontal plane, and a particle of weight w is placed on the other face in a vertical plane through the centre of gravity of the wedge. Show that the particle cannot be in equilibrium on the wedge, if the coefficient of limiting friction between it and the wedge is less than $\tan \alpha$; and determine the tension in a string inclined to the face of the wedge at an angle β and to the horizontal at an angle $\alpha + \beta$, which will just maintain equilibrium.

Will there be any force tending to move the wedge if the string is attached (*a*) to a point connected with the wedge, (*b*) to a point not connected with the wedge? If there is any such force, what is the least coefficient of limiting friction between the wedge and the horizontal plane, that the wedge may not move?

4. A body of mass M has a movable part of mass m; prove that, when the movable part is displaced so that its centre of gravity moves through a distance c, the centre of gravity of the whole is displaced through a distance mc/M in the same direction.

A sheet of paper is in the shape of a rectangle, 9 inches wide by 12 inches long; one of the shorter sides is folded down, so as to lie entirely along one of the longer sides; find the position of the centre of gravity of the whole sheet when thus folded.

5. A ladder AB rests against a smooth vertical wall at A, and is supported by a socket in which its lower end B is placed; the vertical through G (the centre of gravity of the ladder and its load) meets the horizontal through A in K, and the same horizontal cuts the vertical through B in L. Prove that the triangle BKL will serve as a triangle of forces for the weight and the reactions at A, B.

6. A rectangle $ABCD$ in which the sides AB, BC are respectively 8 and 12 cm. is formed of smoothly jointed rods, and is made stiff by a diagonal BD; the frame is hung up from A and a weight of 10 kilogrammes is fastened at C; draw a stress-diagram for the forces in the rods (which are supposed to be light); and hence find the stresses, distinguishing between thrusts and tensions.

1. Horizontal component $= \frac{3W}{2}$, vertical component W, where W is the weight of one rod.

3. Tension $= w \dfrac{\sin(\alpha - \phi)}{\cos(\beta + \phi)}$ if $\mu = \tan\phi$.

(*a*) No force.

(*b*) Coefficient of friction between wedge and ground

$$\frac{w \sin (\alpha - \phi) \cos (\alpha + \beta)}{W \cos (\beta + \phi) + w \cos (\alpha - \phi) \cos (\alpha + \beta)}.$$

4. $3\frac{3}{8}''$ from the unfolded $12''$ side, $4\frac{7}{8}''$ from the unfolded $9''$ side.

6. Thrust in BD 10 kilog. Tensions in AB, CD each $\frac{20}{13}\sqrt{13}$ kilog.

Tensions in BC, AD each $= \frac{30\sqrt{13}}{13}$ kilog.

Intermediate Arts.

1. Find completely the resultant of two unlike parallel forces.

Find the magnitude of the resultant of the following forces acting along the sides of a square $ABCD$, viz.: 11 lb. along DA, 7 lb. along CB, 19 lb. along CD, 5 lb. along BA; and prove that its line of action bisects AD and trisects CD.

2. Show that a system of coplanar forces will be in equilibrium if the algebraic sums of their moments about three points, which are not all in the same straight line, are each zero.

A roller of 21 in. radius lies on the ground; a uniform plank 30 in. long rests flat on the roller with one end on the ground and the other projecting over the roller, the length of the plank being at right angles to the axis of the roller. If the roller and plank are prevented from slipping by a cord 35 in. long, attached to the axle of the roller and to that end of the plank which lies on the ground, prove that the tension of the cord is $\frac{9}{50}$ of the weight of the plank.

3. Having given the masses of a number of coplanar particles and their distances from a fixed line in their plane, obtain a formula for the distance of their centre of gravity from this line.

There are four similar planks, each 12 ft. long, in a pile. The second plank projects 2 ft. beyond the first, the third 3 ft. beyond the second, and the fourth 6 ft. beyond the third ; the sides are flush with each other. Find the centre of gravity of the four planks.

4. Find the force necessary to sustain three movable pulleys, each of weight W, in that system in which each movable pulley hangs in the loop of a separate string, the hanging parts of the strings being vertical and one end of each string being attached to a fixed point.

A, B are two such movable pulleys, each of weight W ; a third pulley, also of weight W, hangs in the loop of a string whose ends are attached one to the axle of A, the other to the axle of B. If B is the upper of the pair A, B, what force must be applied to the free end of the string which passes under B so as to maintain equilibrium ?

[The hanging parts of the strings are supposed vertical.]

5. A framework consists of four light rods, AB, BC, CA, BD, freely jointed at A, B, C, D. The framework is suspended from A and D—two points on the same level 6″ apart. Find the stresses in the rods when a weight of 16 lb. is suspended from C, given that $AB = 8''$, $BC = 15''$, $CA = 17''$, $BD = 10''$, and D and C are on the same side of AB.

6. State and prove the triangle of forces.

The sides BC, CA, AB of a triangle are 7, 5, 3 inches respectively. Find graphically, or otherwise, the magnitude and the inclination to BC of the resultant of the following forces *acting at a point* : 5 lb. in the direction BC, 9 lb. in the direction AC, and 8 lb. in the direction AB.

7. Prove that, if three forces in one plane are in equilibrium, their lines of action meet in a point or are parallel.

A man of weight W is skating on the outside edge ; the friction of the ice at right angles to the blade of the skate is F, the friction in the direction of the blade may be neglected. If the man inclines his body so that the

resultant of all the forces acting on him may be horizontal and pass through his centre of gravity, what is this resultant, and what is the inclination to the vertical of the line joining the skate to the man's centre of gravity?

8. Find the centre of gravity of a uniform triangular lamina; and prove that it coincides with the centre of gravity of three equal particles placed one at the middle point of each side.

$ABCD$ is a square lamina, whose sides are each 9 in.; E, F are points in BC, CD such that EC, CF are each 3 in.; find the centroid of the part $ABEFDA$ of the lamina.

9. Describe the common (Roman) steelyard, and show how to graduate it.

A weighing machine is constructed as follows:—A stiff beam $ABCD$ is pivoted at C, BC being less than CD. Equal rods BF, DE are suspended from B and D, and their ends FE are joined by a rod, of length equal to BD, the rods being freely jointed at B, D, E, F; and a scale pan is attached to the middle point of FE. The counterpoise is a weight P which can slide along AC. If M is the position of P when the machine is in equilibrium without any load, show how to graduate the machine.

1. 30 lb.-wt.

3. $10\frac{1}{2}$ ft. from free end of first plank, and at the height of the upper surface of second plank. 4. $\frac{9}{8}W$.

5. Thrust in BC = 30 lb.; thrust in AB = 40 lb.; tension in CA = 34 lb.; tension in BD = 50 lb.

6. 12·17 lb., $51\frac{1}{2}°$. 7. F, $\tan^{-1}\frac{F}{W}$.

8. On diagonal AC at a distance $\frac{73}{17}\sqrt{12}''$ (or 6·07″) from A.

PRINTED AT THE BURLINGTON PRESS, FOXTON, NEAR CAMBRIDGE, ENGLAND.

A complete Catalogue of Textbooks published by the University Tutorial Press, and separate Sectional Catalogues in Mathematics and Mechanics; Science; English Language and Literature, Geography, and History; Economics, Philosophy, Education, and Religious Knowledge; Latin and Greek; French; may be had on application to the Publishers.

SELECTED TEXTBOOKS

IN

MATHEMATICS

PUBLISHED BY THE

University Tutorial Press Ld.

25 High St., New Oxford St., W.C.

Arithmetic.

The Tutorial Arithmetic. By W. P. Workman, M.A., B.Sc. *Third Edition.* With or without Answers. **6s. 6d.**

Worked Problems in Higher Arithmetic. By W. P. Workman, M.A., B.Sc., and R. H. Chope, B.A. **3s.**

The School Arithmetic. By W. P. Workman, M.A., B.Sc. *Second Edition.* **5s.** With Answers. **5s. 6d.** Also in Two Parts. Part I. With or without Answers. **3s.** Part II. **3s.** With Answers. **3s. 6d.**

The School Arithmetic (School Certificate Edition). By W. P. Workman, M.A., B.Sc. **4s. 6d.** With Answers. **5s.**

The Junior Arithmetic. By R. H. Chope, B.A. *Fifth Edition.* **4s.** With Answers. **4s. 6d.**

Test Questions in Junior Arithmetic. Edited by A. G. Cracknell, M.A., B.Sc., F.C.P. **1s. 6d.** With Answers. **1s. 9d.**

Preliminary Arithmetic. By Augustus Barraclough, M.A. **2s.** With Answers. **2s. 6d.**

Clive's Class Arithmetic. Edited by Wm. Briggs, LL.D., M.A., B.Sc., F.R.A.S. **1s. 6d.** With Answers. **2s.**

Arithmetic—*continued.*

The Primary Arithmetic. Edited by Wm. Briggs, LL.D., M.A., B.Sc., F.R.A.S. *Third Edition.* In Three Parts. Part I. **1s.** Parts II. and III. **1s. 6d.** each. With Answers; each Part **2d.** extra.

Algebra.

The Tutorial Algebra. Advanced Course. By William Briggs, LL.D., M.A., B.Sc., F.R.A.S., and G. H. Bryan, Sc.D., F.R.S. *Fourth Edition.* **8s. 6d.**

A Middle Algebra. (Contained in *The Tutorial Algebra.*) By Dr. William Briggs and Dr. G. H. Bryan. **5s.**

The School Algebra. By A. G. Cracknell, M.A., B.Sc., F.C.P. With or without Answers. **7s. 6d.** Also in Two Parts. Part I. **4s.** Part II. **4s.**

The School Algebra (Matriculation Edition). By A. G. Cracknell, M.A., B.Sc., F.C.P. *Second Edition.* With or without Answers. **6s. 6d.**

Junior Algebra. By A. G. Cracknell, M.A., B.Sc., and Augustus Barraclough, M.A. **3s. 6d.** With Answers. **4s.**

Graphs: The Graphical Representation of Algebraic Functions. By C. H. French, M.A., and George Osborn, M.A. *Second Edition.* **2s.**

Matriculation Graphs. (Contained in *Deakin's Elementary Algebra.*) By C. H. French, M.A., and George Osborn, M.A. *Second Edition.* **1s. 3d.**

Bookkeeping.

Practical Lessons in Bookkeeping. Adapted to the requirements of the Civil Service, Royal Society of Arts, London Chamber of Commerce, etc. By T. C. Jackson, B.A., LL.B., and Wade Hustwick, F.S.A.A. *Sixth Edition.* **5s.** Exercise Books ruled Ledger, Cash, or Journal. **6d.** each.

First Lessons in Bookkeeping (formerly entitled *The Principles of Accounts*). Being the principles of accounts as treated in the first part of *Practical Lessons in Bookkeeping.* By T. C. Jackson, B.A., LL.B., and Wade Hustwick, F.S.A.A. **3s.**

Junior Bookkeeping. By T. C. Jackson, B.A., LL.B. *Second Edition.* **2s. 6d.**

Geometry.

Coordinate Geometry. By J. H. Grace, M.A., F.R.S., and F. Rosenberg, M.A., B.Sc. **6s. 6d.**

The Right Line and Circle (Coordinate Geometry). By Wm. Briggs, LL.D., M.A., B.Sc., F.R.A.S., and G. H. Bryan, Sc.D., F.R.S. *Third Edition.* **5s.**

Mensuration and Spherical Geometry. By Wm. Briggs, LL.D., M.A., B.Sc., F.R.A.S., and T. W. Edmondson, Ph.D., B.A. *Third Edition.* **4s. 6d.**

Euclid. By Rupert Deakin, M.A. Books I.-IV. With an Introductory Course of Practical Geometry and sections on Similar Figures and Geometrical Solids. *Third Edition.* **4s.** Books V., VI., and XI. **3s.**

Deductions in Geometry. Riders and Practical Problems, with full solutions. By T. W. Edmondson, Ph.D., B.A. **3s. 6d.**

Geometry, Theoretical and Practical. By W. P. Workman, M.A., B.Sc., and A. G. Cracknell, M.A., B.Sc.

Part I. (Euclid, I. ; III., 1-34 ; IV., 1-9.) **4s.**
Part II. (Euclid, II. ; III., 35-37 ; IV., 10-16 ; VI.) **3s.**
Part III. (Euclid, XI.) **2s.** ; Paper Covers, **1s.**

Intermediate Geometry: being the Geometry of Similar Figures, Space, and Solids. By A. G. Cracknell, M.A., B.Sc., F.C.P., and G. F. Perrott, M.A. **4s.**

The New Matriculation Geometry. By A. G. Cracknell, M.A., B.Sc., F.C.P., and G. F. Perrott, M.A. *Second Edition.* **4s. 6d.**

School Certificate Geometry. By A. G. Cracknell, M.A., B.Sc., F.C.P., and G. F. Perrott, M.A. **4s. 6d.**
[*In preparation.*

School Geometry (Matriculation Edition). By W. P. Workman, M.A., B.Sc., and A. G. Cracknell, M.A., B.Sc., F.C.P. Adapted from *Geometry, Theoretical and Practical. Second Edition.* **4s. 6d.**

Introduction to School Geometry. By W. P. Workman, M.A., B.Sc., and A. G. Cracknell, M.A., B.Sc., F.C.P. **1s. 6d.**

Junior Geometry. By A. G. Cracknell, M.A., B.Sc., F.C.P. **3s. 6d.**

Preliminary Geometry. By F. Rosenberg, M.A., B.Sc. **3s.**

Trigonometry.

The Tutorial Trigonometry. By Wm. Briggs, LL.D., M.A., B.Sc., F.R.A.S., and G. H. Bryan, Sc.D., F.R.S. *Third Edition.* **5s.**

Junior Trigonometry. By Dr. Wm. Briggs and Dr. G. H. Bryan. **4s.**

Mathematics—General.

Groundwork of Calculus. By Wm. Hunter, M.A., B.Sc. **5s. 6d.**

Clive's Mathematical Tables. Edited by A. G. Cracknell, M.A., B.Sc., F.C.P. **2s. 3d.**

A First Course in Mathematics for Technical Students. By P. J. Haler, M.B.E., M.Sc., A.M.I.Mech.E., and A. H. Stuart, Ph.D., B.Sc. *Second Edition.* **2s.6d.**

A Second Course in Mathematics for Technical Students. By P. J. Haler, M.B.E., M.Sc., A.M.I.Mech.E., and A. H. Stuart, Ph.D., B.Sc. **4s. 6d.**

Mechanics.

The Tutorial Dynamics. By Dr. Wm. Briggs and Dr. G. H. Bryan. *Second Edition.* **5s.**

The Tutorial Statics. By Dr. Wm. Briggs and Dr. G. H. Bryan. *Fourth Edition.* **5s.**

Intermediate Hydrostatics. By Dr. Wm. Briggs and Dr. G. H. Bryan. **5s.**

Matriculation Mechanics. By Dr. Wm. Briggs and Dr. G. H. Bryan. *Fifth Edition.* **5s.**

Matriculation Hydrostatics (Contained in *Intermediate Hydrostatics*). By Dr. Wm. Briggs and Dr. G. H. Bryan. *Second Edition.* **3s. 6d.**

Matriculation Mechanics and Hydrostatics. By Dr. Wm. Briggs and Dr. G. H. Bryan. **7s. 6d.**

Junior Mechanics. By F. Rosenberg, M.A., B.Sc. **4s.**